Asymptotic Expansions for the Heat Kernel and the Trace of a Stochastic Geodesic Flow

Sergio Albeverio

Institut f. Angew. Mathematik

Universität Bonn

Sergio.Albeverio@uni-bonn.de

CERFIM, Locarno

Astrid Hilbert

MSI

Växjö Universitet, Sweden

Astrid.Hilbert@vxu.se

Vassily Kolokoltsov

Department of Statistics

University of Warwick

v.Kolokoltsov@warwick.ac.uk

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Abstract

We analyze the asymptotic behaviour of the heat kernel defined by a stochastically perturbed geodesic flow on the cotangent bundle of a Riemannian manifold for small time and small diffusion parameter. This extends WKB-type methods to a particular case of a degenerate Hamiltonian. We derive uniform bounds for the solution of the degenerate Hamiltonian boundary value problem for small time. From this equivalence of solutions of the Hamiltonian equations and the corresponding Hamilton Jacobi equation follows. The results are exploited to derive two sided estimates and multiplicative asymptotics for the heat kernel and the trace.

1   Introduction

The problem of finding Gaussian bounds for the heat kernel and of deriving asymptotic expansions for non degenerate diffusions has been studied by various researchers. Expansions for the stochastic flow and stochastic variational calculus have been exploited by [Mol75], [Dav88], [ABdMBB95], [KS87], [BA89], and [Léa94]. For hypoelliptic operators satisfying a uniform weak Hörmander hypothesis uniform in

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space estimates for the heat kernel have been given, e.g., by Léandre. For Gaussian hypoelliptic generators D. Manankiandrianana [CME81] and M. Chaleyat, L. Elie [Man79] have classified the structure of the corresponding processes, derived the associate action functionals, and the associated subriemannian metrics.

In [Mas85] Maslov observed that for a class of pseudodifferential equations (called tunnel equations) on $\mathbb{R}^n$ which includes the example of elliptic second order equations asymptotic bounds for the heat kernel can be established for small values of a parameter h (Planck's constant). Results using WKB-type methods may be found e.g. in [Mas72], [Mas85], [MF81], [KM97], [Kol00]. In this paper we modify the WKB-type method by introducing a change of variable which adjust the different time scales of the variables. We derive bounds for the kernel of degenerate diffusions associated with the geodesic flow on a compact d-dimensional Riemannian manifold M with metric g which is subject to a stochastic perturbation in the tangent space variable. In local coordinates the diffuion on the cotangent bundle T^*M is given by.

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \mathcal{L}u = \frac{h}{2}\left(g_{ij}(x)\frac{\partial^2 u}{\partial y_i \partial y_j}\right) + \left(G(x)y, \frac{\partial u}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x}(G(x)y, y), \frac{\partial u}{\partial y}\right) \quad (1)\\
&= : h^{-1}H\left(x, y, -h\frac{\partial}{\partial x}, -h\frac{\partial}{\partial y}\right)u
\end{aligned}
$$

with initial data $u(0, x, y, x_0, y^0) = \delta(x-x_0)\delta(y-y^0)$ for x and y in a neighbourhood of the point $(x_0, y^0) \in T^*M$, $h > 0$ is a (small) parameter $g_{ij}(x)$ is the expression of the metric in local coordinates with inverse $G^{ij}(x)$. The summation convention over indices is used. The pseudodifferential operator $H(x, y, -\frac{\partial}{h\partial x}, -\frac{\partial}{h\partial y})$ defines a Hamiltonian function $H(x, y, p, q)$ the arguments of which can be divided into the variables $x, y \in \mathbb{R}^d$ and their respective conjugate variables i.e. the momenta p and q. In local coordinates the Hamiltonian takes the form

$$
H(x, y, q, p) = \frac{1}{2}\left(g(x)q, q\right) - \left(G(x)y, p\right) + \frac{1}{2}\left(\frac{\partial}{\partial x}(G(x)y, y), q\right) . \quad (2)
$$

It is possible to write a stochastic differential equation associated with the operator $\mathcal{L}$. One way is to introduce an imbedding r of the manifold M into a large Euclidian space $\mathbb{R}^N$ (such an imbedding always exists). In local coordinates the corresponding stochastic system will have the form

$$
\begin{aligned}
dx_i &= \frac{\partial H}{\partial y_i}dt\\
dy_i &= -\frac{\partial H}{\partial x_i}dt + \frac{\partial}{\partial x_i}(\gamma, d\omega) \quad (3)
\end{aligned}
$$

where $(x.y) \in T^*M$, ω is standard Brownian Motion in $\mathbb{R}^d$ and $(\cdot, \cdot)$ denotes the standard scalar product in $\mathbb{R}^N$. The system describes a Newtonian particle corresponding to a Hamiltonian system perturbed by a Gaussian White Noise force "$\dot{w}$"

with the friction $\frac{1}{2}\gamma_i^{kl} y_k y_l$ see e.g. [Nel67], [AHZ92], [MS00] for the flat case or [Joe78] for a covariant definition. In the sequel neglecting friction we shall restrict ourselves to the example of the geodesic flow on the cotangent bundle T^*M of a Riemannian manifold M. In this case the Hamiltonian corresponding to the deterministic part of (3) is of the form $f(x) = \frac{1}{2}(G(x)y, y)$ where $G(x) = g^{-1}(x)$ is smooth by definition, see also (2).

The content of the paper is organized in five sections. The second outlines the general structure of an invariant diffusion. In the subsequent one we study the underlying Hamiltonian dynamics of the stochastic geodesic flow. This results in the uniform estimate (19) and in the existence of a local diffeomorphic flow on the cotangent bundle. In Section 4 we explain the WKB-method for imaginary time and the change of variables which compensates for the degeneracy and allows for a closed system of algebraic equations for the coefficients of an asymptotic expansion for the solution of (1) in powers of h. The remaining part of this section is dedicated to the existence of a multiplicative error. The paper is concluded by discussing an asymptotic expansion for the trace formula for the generator corresponding to a stochastic geodesic flow (1). The coefficients of the expansion are geometric invariants of the manifold.

2   Geometric background for the system under consideration

Let us introduce a diffusion process on the cotangent bundle of a compact Riemannian manifold M with metric g which in local coordinates has the form

$$
\begin{aligned}
h\frac{\partial}{\partial t}u &= \frac{h^2}{2}g_{ij}(x)\frac{\partial^2}{\partial y_i \partial y_j}u + h(a^i(x) + \alpha^{ij}(x)y_j)\frac{\partial u}{\partial x^i} \\
&\quad + h(b_i(x) + \beta_i^j(x)y_j + \frac{1}{2}\gamma_i^{jl}(x)y_j y_l)\frac{\partial u}{\partial y_i} - V(x,y)u
\end{aligned}
\tag{4}
$$

where h is a strictly positive parameter, the $d \times d$-matrix $g(x) = (g_{ij}(x))$ is positive definite, α, β, γ are $d \times d$-matrices, α being non-degenerate, a, b are uniformly bounded in $x \in \mathbb{R}^d$ and V is a polynomial of degree 4 in $y \in \mathbb{R}^d$ with coefficients depending on $x \in \mathbb{R}^d$ bounded from below. This diffusion defines a regular degenerate diffusion of rank one in the sense of [Kol00]. In this book there are given necessary and sufficient conditions for this diffusion to be invariant.
Let us show that the pseudodifferential operator in (1) is an invariant degenerate diffusion. In order to study invariance under change of coordinates in the manifold, we recall that changing the variable $x \to \tilde{x}$ for the diffusion (1) induces a change of

the momentum variables according to the rule $\tilde{y} = y\frac{\partial x}{\partial \tilde{x}}$. This implies e.g.

$$\frac{\partial \tilde{y}_k}{\partial y_j} = \frac{\partial x^j}{\partial \tilde{x}^k}, \quad \frac{\partial u}{\partial x^i} = \frac{\partial u}{\partial \tilde{x}^\ell}\frac{\partial \tilde{x}^\ell}{\partial x^i} + \frac{\partial u}{\partial \tilde{y}_\ell}\frac{\partial \tilde{y}_\ell}{\partial x^i} \tag{5}$$

$$\frac{\partial u}{\partial y_i} = \frac{\partial u}{\partial \tilde{y}_\ell}\frac{\partial \tilde{y}_\ell}{\partial y_i}, \quad \frac{\partial^2 u}{\partial y_i \partial y_j} = \frac{\partial^2 u}{\partial \tilde{y}_k \partial \tilde{y}_\ell}\frac{\partial \tilde{y}_\ell}{\partial y_j}\frac{\partial \tilde{y}_k}{\partial y_i}. \tag{6}$$

for $i, j, k = 1, \ldots, d$. An immediate consequence of (5) and (6) is that the terms of different orders do not mix and hence may be considered separately. Then the invariance of equation (4) follows from direct calculation using the relations (5) and (6) and the fact that the metric tensor transforms according to $\tilde{g}_{ij}(\tilde{x}) = g_{ij}(x)\frac{\partial x^j}{\partial \tilde{x}^\ell}\frac{\partial x^i}{\partial \tilde{x}^k}$. The calculations in this paper will be carried out in normal coordinates around x_0. We are using the definition given in [CFKS87] for which $x = 0$ and for which the matrix associated with the Riemannian metric satisfies $\det g(x) = 1$, which can be achieved by a change of variables and the expansion

$$g_{ij}(x) = \delta_i^j + \frac{1}{2}g_{ij}^{kl}x^k x^l + O(|x|^3). \tag{7}$$

These conditions imply

$$\sum_{i=1}^{n} g_{ii}^{kl} = 0 \quad \text{for} \quad 1 \le k, l \le d$$

and the following representation for the Gausssian (scalar) curvature

$$R = \sum_{i,k} g_{ik}^{ik}.$$

For the corresponding inverse matrix $G(x) = g^{-1}(x)$ and its derivative we get the following Taylor approximations

$$G^{ij}(x) = \delta_i^j - \frac{1}{2}g_{kl}^{ij}x^k x^l + O(|x|^3) \quad \text{and} \quad \Gamma_k^{ij}(x) = -g_{kl}^{ij}x^\ell + O(|x|^3). \tag{8}$$

Here $\Gamma_k^{ij}(x) := \frac{\partial G_{ij}}{\partial x^k}(x)$ are the coefficients of the Levy Civita connection in local coordinates. For symmetry reasons we also introduce the notation $\gamma_{i\ell j} = \frac{\partial g_{i\ell}}{\partial x^j}$ which will be used in the next section.

3 Hamiltonian Dynamics

The Hamiltonian system corresponding to (1) is given by

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = -G^{ij}(x)y_j \tag{9}$$

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$$\dot{y}_i = \frac{\partial H}{\partial q^i} = \frac{1}{2}\frac{\partial}{\partial x^i}(G(x)y, y) + \frac{1}{2}g_{ij}(x)q^j = \frac{1}{2}\Gamma_i^{kj}(x)y_k y_j + \frac{1}{2}g_{ij}(x)q^j$$

$$\dot{q}^i = -\frac{\partial H}{\partial y_i} = G^{ij}(x)p_j - \left(\frac{\partial}{\partial x}G^{ij}(x)y_j, q\right) = G^{ij}(x)p_j - \Gamma_k^{ij}(x)y_j q^k$$

$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = \left(\frac{\partial}{\partial x^i}G(x)y, p\right) - \frac{1}{2}\left(\frac{\partial}{\partial x^i}\frac{\partial}{\partial x}(G(x)y, y), q\right) - \frac{1}{2}\left(\frac{\partial}{\partial x^i}g(x)q, q\right)$$

with initial condition $x(0) = x_0$, $y(0) = y^0$, $q(0) = q_0$, $p(0) = p^0$. The corresponding flow will be denoted by (X, Y, Q, P).

Hypothesis 3.1. *Let there exist constants k, t_0 and c_0 such that for all $c \in (0, c_0]$ and $t \in (0, t_0]$ the solution to (9) exists on $[0, t]$ whenever the initial data y^0, q_0, p^0 are satisfying $|y^0| \le \frac{c}{t}$, $|q_0| \le \frac{c^2}{t^2}$, $|p^0| \le \frac{c^3}{t^3}$ and x_0 may be chosen freely.*

Proposition 3.1. *Under the hypothesis 3.1 the components of the flow (X, Y, P, Q) started at (x_0, y^0, p^0, q_0) satisfy*

$$|x(t) - x_0| \le kt(1 + \frac{c}{t}), \quad |y(t) - y^0| \le kt(1 + \frac{c^2}{t^2})$$

$$|q(t) - q_0| \le kt\left(1 + \frac{c^3}{t^3}\right), \quad |p(t) - p^0| \le kt(1 + \frac{c^4}{t^4}).$$

Proof. The proof uses Taylor expansion and is a direct consequence of the fact that the matrix g is a bounded function of the variable x (c.f. also Proposition 2.3.1 in [Kol00]). $\square$

If we associate with the variables x, y, q, p the degree $0, 1, 2, 3$, respectively each right hand side of (9) is of specific degree varying between 1 for $\dot{x}$ and 4 for $\dot{p}$. The degree is related to the power of $\frac{1}{t}$ which is crucial for the iteration procedure giving the expansions intrinsic to the WKB-method.

In the subsequent proposition we show that the solution (x, y, q, p) of (9) starting at (x_0, y^0, q_0, p^0) can be represented in terms of the solution $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q})$ solving (9) with initial value $(x_0, y^0, 0, 0)$. For reason of space we shall skip the arguments of the functions $g(x)$, $G(x)$ and of their derivatives in future.

Proposition 3.2. *Under the assumptions of Proposition 3.1 we have:*

$$x = \tilde{x} - \frac{1}{2}t^2 q_0 - \frac{1}{6}t^3[p^0 + (\Omega' y^0) q_0] + \delta^4$$

$$y = \tilde{y} + tq_0 + \frac{1}{2}t^2[p^0 + (\Omega y^0) q_0] + \frac{1}{t}\delta^4$$

$$q = \tilde{q} + t[G(x_0)p^0 + (\Gamma y^0)q_0]\delta^4 + \frac{1}{2}t^2[(\Lambda y^0)p^0 + (\Lambda'(y^0)^2)q_0 + \Lambda''(q_0)^2] + \delta$$

$$p = \tilde{p} + t\delta^4.$$

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where the error term δ is of the order $O(t+c)$. Moreover, $\tilde{x} = x_0 - y_0 t + O(t^2)$, $\tilde{y} = y^0 + O(t)$, $\tilde{p} = O(t)$, $\tilde{q} = O(t)$, and

$$
\begin{aligned}
(\Gamma y_0)_i^k &= \Gamma_i^{jk} y_j^0 = \frac{\partial G^{jk}}{\partial x^i} y_j^0, & \left(\tilde{\Gamma} y^0, y^0\right)_{ik} &= \tilde{\Gamma}_{ik}^{\ell j} y_\ell^0 y_j^0 = \frac{\partial \Gamma_i^{\ell j}}{\partial x^k} y_\ell^0 y_j^0 \\
(\Omega y_0)_{i\ell} &= \left[g_{ij} \Gamma_\ell^{jm} g_{j\ell} - \gamma_{i\ell j} G^{jm} \right] (x_0) y_m^0, \\
\left(\Omega' y^0\right)_{ik} &= \Gamma_i^{j\ell} g_{jk} y_\ell^0 - \gamma_{ikm} G^{m\ell} y_\ell^0 - g_{ij} \Gamma_k^{j\ell} y_\ell^0 \\
(\Lambda y_0)^{ik} &= -\Gamma_m^{ik} G^{m\ell} y_\ell^0 + G^{im} \Gamma_m^{k\ell} y_\ell^0 - G^{mk} \Gamma_m^{ij} y_j^0, \\
\left(\Lambda'(y_0)^2\right)_k^i &= -\frac{1}{2} G^{im} \tilde{\Gamma}_{mk}^{\ell j} y_\ell^0 y_j^0 - \tilde{\Gamma}_{mk}^{ij} G^{m\ell} y_\ell^0 y_j^0 - \frac{1}{2} \Gamma_k^{ij} \Gamma_j^{m\ell} y_\ell^0 y_m^0 - \Gamma_m^{ij} \Gamma_k^{m\ell} y_j^0 y_\ell^0. \\
(\Lambda'')_{k\ell}^i &= -\frac{1}{2} G^{im} \gamma_{k\ell m} - \frac{1}{2} g_{m\ell} \Gamma_k^{im}
\end{aligned}
$$

Proof. For the proof we expand the components $p - \tilde{p}$, $q - \tilde{q}$, $y - \tilde{y}$, $x - \tilde{x}$ in Taylor series with respect to time. For the time derivatives of the expansion we insert the explicit expressions given by the system of Hamilton equations (9). Due to linearity of differentiation it suffices to study the derivatives of $y(t)$ and insert $q_0 = 0$ and $p^0 = 0$ to retrieve the derivatives of $\tilde{y}(t)$. We begin with the function $\Delta p_i(t) = p_i(t) - \tilde{p}_i(t)$ because it is of highest degree and hence determines the degree to be considered for all other components. We have $\Delta p_i(t) = p_i^0 + \int_0^t \dot{\Delta} p_i(\tau) \, d\tau$. We directly find:

$$
\begin{aligned}
\Delta p_i(t) &= p_i^0 + \int_0^t \Gamma_i^{k\ell} y_\ell p_k - \frac{1}{2} \tilde{\Gamma}_{mi}^{k\ell} y_\ell y_m q^k - \frac{1}{2} \gamma_{ijk} q^j q^k \, d\tau & (10) \\
&= p_i^0 + \int_0^t L(y^0 p^0, (y^0)^2 q_0) + O(\tau^2) \, d\tau \, . & (11)
\end{aligned}
$$

Here L is a linear function of degree 4 in its arguments with bounded coefficients depending on $x(t) \in \mathbb{R}^d$. For $\dot{\tilde{p}}$ the term corresponding to L vanishes except for the uniform error $O(t^3)$. Inserting the estimates given in Proposition 3.1 in the integrand gives the desired result. For the function $\Delta q_i(t) = q_i(t) - \tilde{q}_i(t) = q_0^i + t \dot{\Delta q}^i{}_0 + \int_0^t (t - \tau) \ddot{\Delta q}^i(\tau) \, d\tau$ we set off analogously:

$$
\begin{aligned}
\dot{q}^i(\tau) &= G^{ij} p_j - \Gamma_\ell^{ij} y_j, q^\ell \text{ in particular } \dot{q}_0^i(\tau) = G^{ij} p_j^0 - \Gamma_\ell^{ij} y_j^0, q_0^\ell \\
\ddot{q}^i(\tau) &= \frac{\partial G^{ij}}{\partial x} \dot{x} y_j + G^{ij} \dot{p}_j - \left(\frac{\partial^2 G^{ij}}{\partial x \partial x} \dot{x} y_j, q \right) - \left(\frac{\partial G^{ij}}{\partial x} \dot{y}_j, q \right) - \left(\frac{\partial G^{ij}}{\partial x} y_j, \dot{q} \right)
\end{aligned}
$$

Direct calculation gives a long expression the structure of which becomes clear by inserting the notation in the proposition.

$$
\ddot{q}(\tau) = (\Lambda y) p + (\Lambda' y^2) q + \Lambda'' q^2 = (\Lambda y) p^0 + (\Lambda' y^2) q_0 + \Lambda'' q_0^2 + O(t) \, .
$$

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A direct consequence is that $\tilde{q}^i\dot{}(0)$ and $\tilde{q}^i\ddot{}(0)$ vanish except for an error of order $O(t)$. We find using the estimates of Proposition 3.1 under the integral

$$
\begin{aligned}
\Delta q^i(t) &= q_0^i + tL(y^0 q_0, p^0) + \int_0^t (t-\tau)L(y(\tau)p(\tau), y^2(\tau)q(\tau), q^2(\tau))\, d\tau \\
&= q_0^i + tL(y^0 q_0, p^0) + t^2 L(y^0 p^0, (y^0)^2 q_0, q_0^2) + O(t^3) \qquad (12)
\end{aligned}
$$

where we proceeded as above. In order to achieve an analogous representation for the function $y(t)$ the starting equation is given by

$$
\Delta y_i(t) = y_i(t) - \tilde{y}_i(t) = y_i^0 + t\Delta\dot{y}_i{}^0 + \frac{1}{2}t^2\Delta\ddot{y}_i{}^0 + \int_0^t (t-\tau)^2 \Delta(y_i)^{(3)}(\tau)\, d\tau\ . \quad (13)
$$

The first order derivatives are explicitly given in (9). For $2\ddot{y}_i(t)$ we have

$$
\begin{aligned}
\Big(-\tilde{\Gamma}_{ik}^{j\ell}G^{km}y_m y_j \;&+\; \Gamma_i^{j\ell}(\Gamma_j^{km}y_k y_m - g_{jm}q^m)\Big)y_\ell - \gamma_{i\ell j}G^{jm}y_m q^\ell + g_{ij}(G^{j\ell}p_\ell - \Gamma_\ell^{jm}y_m q^\ell) \\
&= (\Omega''y^2)_i^\ell y_\ell - (\Omega'y)_{i\ell}q^\ell + g_{ij}G^{j\ell}p_\ell = L(y^3, yq, p)
\end{aligned}
$$

with $(\Omega''(y_0)^2)_i^\ell = -\tilde{\Gamma}_{ik}^{j\ell}G^{km}y_j^0 y_m^0 + \Gamma_i^{jm}\Gamma_j^{\ell k}y_m^0 y_k^0$. The corresponding term for $\tilde{\ddot{y}}(0)$ does not vanish but the terms cancel. Here L is a linear function of degree 3 in its arguments with coefficients depending on $x(t) \in \mathbb{R}^d$. In the sequel we make use of the fact that we may change the names of the indices. This way we get e.g. the equality $\Gamma_i^{j\ell}g_{jk}q^k y_\ell = \Gamma_i^{jm}g_{j\ell}q^\ell y_m$. Furthermore $2y_i^{(3)}(t)$ is of the form

$$
-\frac{\partial \tilde{\Gamma}_{im}^{j\ell}}{\partial x_n}\dot{x}^n G^{mk}y_j y_\ell y_k - 2\tilde{\Gamma}_{im}^{j\ell}G^{mk}\dot{y}_j y_\ell y_k - \tilde{\Gamma}_{im}^{j\ell}\Gamma_n^{mk}\dot{x}^n y_j y_\ell y_k - \tilde{\Gamma}_{im}^{j\ell}G^{mk}y_j y_\ell \dot{y}_k
$$

$$
+\tilde{\Gamma}_{in}^{j\ell}\dot{x}^n(\Gamma_j^{km}y_\ell y_k + g_{j\ell}q^\ell)y_m + \Gamma_i^{j\ell}(\tilde{\Gamma}_{jn}^{km}\dot{x}^n y_m y_k + 2\Gamma_j^{km}\dot{y}_m y_k - \gamma_{jmk}\dot{x}^k q^m - g_{jm}\dot{q}^m)y_\ell
$$

$$
+\Gamma_i^{j\ell}(\Gamma_j^{km}y_m y_k + g_{jm}q^m)\dot{y}_\ell - \frac{\partial\gamma_{ijm}}{\partial x^k}G^{m\ell}\dot{x}^k y_\ell q^j - \gamma_{ijm}(\Gamma_k^{m\ell}\dot{x}^k y_\ell q^j + G^{m\ell}(\dot{y}_\ell q^j + y_\ell \dot{q}^j))
$$

$$
+\gamma_{ijk}\dot{x}^k(G^{j\ell}p_\ell - \Gamma_\ell^{jm}y_m q^\ell) + g_{ij}\left(\Gamma_k^{j\ell}\dot{x}^k p_\ell + G^{j\ell}\dot{p}_\ell - \tilde{\Gamma}_{\ell k}^{jm}\dot{x}^k y_m q^\ell - \Gamma_\ell^{jm}(\dot{y}_m q^\ell + y_m \dot{q}^\ell)\right)
$$

which constitutes a linear function $L(y^4, y^2 q, q^2, yp)$ of degree 4 in its arguments with coefficients depending on $x(t) \in \mathbb{R}^d$. For the difference corresponding to y^4 and $\tilde{y}^4$ we give a direct method. Expanding the tensor with respect to x and $\tilde{x}$ and applying that we are working in Normal coordinates we are left with an error of the order $O(t^5)$ for the difference of these terms. For the integrand the estimates given in Proposition 3.1 apply. We find

$$
y(t) = \tilde{y}(t) + tg(x)q_0 + \frac{1}{2}t^2[g(x)G(x)p^0 + (\Omega y)q_0]\ .
$$

As can be read from the preceding equation the higher order terms corresponding to $\ddot{y}(t)$, $y_i^{(3)}(t)$ adequately become part of the error term δ. In order to achieve an

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analogous representation for the function $\Delta x = x - \tilde{x}$ we start from the equation $\Delta x^i(t) = x_0^i + t\Delta\dot{x}_0^{\,i} + \frac{1}{2}t^2\Delta\ddot{x}_0^{\,i} + \int_0^t (t-\tau)^3 \Delta x^{i(4)}(\tau)\,d\tau$. Here the values of the derivatives are determined by inserting into

$$
\begin{aligned}
\ddot{x}^i(t) &= \Gamma_m^{ij} G^{m\ell} y_\ell y_j - G^{ij}\dot{y}_j = L(y^2, q)\\
x^{i(3)}(t) &= -(\tilde{\Gamma}_{mn}^{ij} G^{m\ell} + \Gamma_m^{ij}\Gamma_n^{m\ell})G^{nk} y_k y_\ell y_j + \Gamma_m^{ij}G^{m\ell}(\dot{y}_\ell y_j + 2y_\ell \dot{y}_j) - G^{ij}\ddot{y}_j\\
&= L(y^3, yq, p)\\
x^{i(4)}(t) &= -\left(\frac{\partial \tilde{\Gamma}_{mn}^{ij} G^{m\ell}}{\partial x^\nu} + \tilde{\Gamma}_{mn}^{ij}\tilde{\Gamma}_{n\nu}^{m\ell} + \tilde{\Gamma}_{m\nu}^{ij}\Gamma_n^{m\ell} + \Gamma_m^{ij}\tilde{\Gamma}_{n\nu}^{m\ell} \right) G^{nk}G^{\nu h} y_h y_k y_\ell y_j\\
&\quad - (\tilde{\Gamma}_{mn}^{ij} G^{m\ell} + \Gamma_m^{ij}\Gamma_n^{m\ell})G^{nk}y_k(\dot{y}_\ell y_j + 2y_\ell \dot{y}_j) + \Gamma_m^{ij}G^{m\ell}(\ddot{y}_\ell y_j + 3\dot{y}_\ell \dot{y}_j + 3y_\ell \ddot{y}_j)\\
&\quad - G^{ij} y_j^{(3)} = L(y^4, y^2 q, q^2, yp).
\end{aligned}
$$

Here $L(y^2, q)$, $L(y^3, yq, p)$ and $L(y^4, y^2 q, q^2, yp)$ stand for different linear functions of degree 2, 3 and 4, respectively. The explicit expressions of these functions follow by inserting the previous results on $\dot{x}$, $\ddot{x}$, and $x^{(3)}$. For the difference of the homogenous terms $x^4 \, \tilde{y}^4$ we proceed as above leaving a uniform error in time in the integrand . All together we find

$$
x^i(t) = \tilde{x}^i - \frac{t^2}{6}G^{ij}g_{ij}p^0 + \frac{t^3}{6}\left(2\frac{\partial \Gamma_m^{ij}}{\partial x^k}G^{kn}G^{m\ell} + \frac{\partial \Gamma_m^{i\ell}}{\partial x^k}G^{kn}G^{mj} \right) G^{jn} y_n y_\ell p_n^0 \ . \tag{14}
$$

This concludes the proof of 3.2. $\qquad\square$

Using the expression for the solution of the Hamilton equation given in Proposition 3.2 above we find the approximations for the corresponding derivatives w.r. to the initial data q_0 and p^0 by plain derivation.

Corollary 3.1. *Under the assumptions of 3.1 the asumptotic behaviour of the partial derivatives of the components of the solution to (9) is given by*

$$
\begin{aligned}
\frac{\partial X}{\partial p^0} &= \frac{1}{6}t^3(1d + \delta), & \frac{\partial X}{\partial q_0} &= -\frac{1}{2}t^2(1d + \frac{t}{3}(\Omega'y^0) + \delta^2)\\
\frac{\partial Y}{\partial p^0} &= \frac{1}{2}t^2(1d + \delta), & \frac{\partial Y}{\partial q_0} &= t(1d + \frac{t}{2}(\Omega y^0) + \delta^2)\\
\frac{\partial P}{\partial p^0} &= 1d + \delta, & \frac{\partial P}{\partial q_0} &= \frac{\delta^2}{t}, \quad \frac{\partial Q}{\partial p^0} = t \cdot \delta, \quad \frac{\partial Q}{\partial q_0} = 1 + t\left(\frac{\partial}{\partial x}G(x_0) \right)y^0 + \delta^2
\end{aligned}
$$

The bounds for the derivatives given in Proposition 3.2 (and hence in its Corollary) can be generalized to higher order derivatives in p^0 and q_0 which will be used later. Let us start with the y-component of the solution to the Hamilton equation. Instead of the second order Taylor polynomial (13) we start off with a fifth order polynomial

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since the second order derivative w.r. to p^0 needs to be carried out explicitly. This corresponds to $y^{(5)}$ being a linear function L with argument of degree 6. In the proof of Proposition 3.2 the derivatives up to 3rd order are calculated explicitly. Furthermore, up to 4th order the argument of the nth order derivative of x and the $(n-1)^{st}$ order derivative of y are of the same degree.

For the higher order derivative we proceed in the following formal way, in particular, our notation does not distinguish different linear functions L:

$$
\begin{aligned}
y^{(5)}(t) &= \frac{d}{dt} L(y^4, y^2 q, q^2, yp) \\
&= \frac{\partial L}{\partial x}(y^4, y^2 q, q^2, yp)\dot{x} + \frac{\partial L}{\partial y}(y^3, yq, p)\dot{y} + \frac{\partial L}{\partial q}(y^2, q)\dot{q} + \frac{\partial L}{\partial p}(y)\dot{p} \\
&= L(y^5, y^3 q, yq^2, y^2 p) + L(y^5, y^3 q, yq^2, y^2 p, qp) \\
&\qquad L(y^3 q, yq^2, y^2 p, qp) + L(y^3 q, yq^2, y^2 p) \\
&= L(y^5, y^3 q, yq^2, y^2 p, qp)
\end{aligned}
\tag{15}
$$

and analogously we find

$$
\begin{aligned}
y^{(6)}(t) &= L(y^6, y^4 q, y^2 q^2, q^3, y^3 p, yqp, p^2) && (16) \\
x^{(6)}(t) &= L(y^5, y^3 q, yq^2, y^2 p, qp) && (17) \\
x^{(7)}(t) &= L(y^6, y^4 q, y^2 q^2, q^3, y^3 p, yqp, p^2) \ . && (18)
\end{aligned}
$$

The algebraic structure behind the procedure is similar to the one studied in [HL01], [NSW85], [BA89] and was developed in the case of a regular diffusion in [Kol00].

Since the coefficients are linear combinations of different orders of derivatives of the functions g and G we may conclude that the coefficients do not vanish simultaneously. Expanding q and p in a Taylor series w.r. to t we find those functions L which explicitly depend on $(q_0)^2$, $q_0 p^0$, and $(p^0)^2$. We are interested in the lowest order in t for which these arguments appear.

As was done in the course of the proof to Corollary 3.1 for the first order derivatives the coefficients of the second order derivatives in q_0 and p^0 of X and Y are bounded given by polynomials in t. This result is summarized in the following corollary.

Corollary 3.2. *Under the assumptions of 3.1 the asymptotic behaviour of the second order partial derivatives of the components of the solution to (9) is given by*

$$
\begin{array}{ccc}
\frac{\partial^2}{\partial q_0^2} X = O(t^4) & \frac{\partial^2}{\partial q_0 p^0} X = O(t^5) & \frac{\partial^2}{\partial (p^0)^2} X = O(t^6) \\
\frac{\partial^2}{\partial q_0^2} Y = O(t^3) & \frac{\partial^2}{\partial q_0 p^0} Y = O(t^4) & \frac{\partial^2}{\partial (p^0)^2} Y = O(t^5) \ .
\end{array}
$$

The result proven in the corollary above can be summarized in the following way:

$$
\frac{\partial (X,Y)^I}{\partial \zeta^{I_1} \ldots \partial \zeta^{I_k}} = O(t^{3k - I - I_1 - \ldots - I_k}) \ .
\tag{19}
$$

where we have $I = 0$ for X and $I = 1$ for Y as well as $I_k = 0$ for q_0 and $I_k = 1$ for p^0 cf. [Kol00].

From Corollary 3.2 follows that the matrix $\frac{\partial(X,Y)}{\partial(p^0,q_0)}$ is non degenerate hence invertible and bounded for each $t \in (0, t_0]$, $t_0 < \infty$. In particular

$$J(t,x,y,x_0,y^0) = det\frac{\partial(X,Y)}{\partial(p,q)}(t,x,y,p,q)_{\substack{p=p^0(t,x,y,x_0,y^0)\\q=q_0(t,x,y,x_0,y^0)}} \tag{20}$$

does not vanish provided the conditions of Proposition 2.1 hold.

Theorem 3.1. *Let the assumptions of Proposition 3.1 be satisfied. i) There exist positive real numbers c and t_0 (uniformly in x_0) such that for all $t \leq t_0$ and $|y^0| \leq \frac{c}{t}$ the mapping $(q_0, p^0) \longmapsto (X, Y)(t, x_0, y^0, q_0, p^0)$ defined on the polydisk $B_{\frac{c^2}{t^2}} \times B_{\frac{c^3}{t^3}}$ is a diffeomorphism onto its image.*
ii) There exists $r > 0$ such that by reducing c and t_0 if necessary, one can assume that the image of the diffeomorphism described in i) contains a poly disc $B_r(\tilde{x}) \times B_{\frac{r}{t}}(\tilde{y})$.

Proof. i) Due to the asymptotic expansion for (X, Y) given in Proposition 3.2 the Jacobian of the mapping $(q_0, p^0) \mapsto (X, Y)(t, x_0, y^0, q_0, p^0)$ does not vanish for $t \in [0, t_0]$, $t_0 > 0$, while the assumptions of Proposition 3.2 hold. Hence the mapping itself is invertible. Moreover, the mapping constitutes a diffeomorphism since we are implicitly given the expansion for the inverse where the t-dependence of the poly disk compensates for the different behaviour in time of the parameters. This finishes the proof of the statement i).
ii) We have to prove that for arbitrary $(x, y) \in B_r(\tilde{x}) \times B_{\frac{r}{t}}(\tilde{y})$ there exists $(q_0, p^0) \in B_{\frac{c^2}{t^2}} \times B_{\frac{c^3}{t^3}}$ such that $(x, y) = (X, Y)(t, x_0, y^0, p^0, q_0)$. This is equivalent to proving the existence of a fixed point as for the non-degenerate case treated in [Kol00]. $\qquad\square$

The projections on the (X, Y)-space of the solutions to (9) are called characteristics of the Hamiltonian H given by (2). Suppose that for each initial value $(x_0, y^0) \in \mathbb{R}^{2n}$, and $t \in [0, t_0]$, $t > 0$, there exists a neighbourhood $\Omega = \Omega(t, x_0, y^0)$ of the origin in the (q, p)-space $\mathbb{R}^{2n}$ such that the mapping $(q_0, p^0) \mapsto (X, Y)(t, x_0, y^0, q_0, p^0)$ is a diffeomorphism from Ω onto its image containing a neighbourhood $D(x_0, y^0)$, then the family

$$\Gamma(x_0, y^0) = \{(X, Y)(t, x_0, y^0, q_0, p^0) | (q_0, p^0) \in \Omega(x_0, y^0)\}, 0 \leq t \leq t_0$$

of solutions is called a field of characteristics starting at (x_0, y^0). Due to Corollary 3.1 and the subsequent arguments this means that under the assumptions of Proposition 3.1 there exists a field of characteristics for each initial point (x_0, y^0) i.e. there exists a smooth function

$$(Q_0, P^0)(t, x, y, x_0, y^0) : (0, t_0] \times D(x_0, y^0) \to \Omega$$

such that
$$(X, Y) \left(t, x_0, y^0, (P^0, Q_0)(t, x, y, x_0, y^0) \right) = (x, y) . \tag{21}$$

Recalling that there exists a field of characteristics for any two points (x_1, y_1) and (x_2, y_2) joint by a characteristic curve of the Hamiltonian H we introduce the function

$$\sigma(t, x_0, y^0, q_0, p^0) = \int_0^t \left((P, Q), (\dot{X}, \dot{Y}) \right)(\tau) - H(X, Y, P, Q)(\tau)d\tau , \tag{22}$$

where the components X, Y, P, Q of the solution to the Hamiltonian system corresponding to H are being evaluated at an intermediate time τ, $0 \le \tau \le t$. Due to Proposition 3.1 This function is diffeomorphic equivalent to

$$S(t, x, y, x_0, y^0) = \int_0^t (Q, P)(\tau)d\gamma - \int_0^t H(\gamma, P, Q)(\tau)d\tau , \tag{23}$$

which is called the *two point function*.

Proposition 3.3. *Under the assumptions of Proposition 3.1 and Theorem 3.1 the function* $S(t, x, y, x_0, y^0)$ *satisfies the Hamilton Jacobi equation*

$$\frac{\partial S}{\partial t} + H(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}) = 0$$

as a function of $t \in (0, t_0]$ *and* $x, y \in D(x_0, y^0)$ *with* $D(x_0, y^0)$ *as above. Moreover, we have*

$$\begin{aligned}
\frac{\partial S}{\partial x}(x, y) &= p(t, x, y), \quad \frac{\partial S}{\partial y}(t, x, y) = q(t, x, y). \\
\frac{\partial S}{\partial x_0}(x, y), &= -p^0(t, x, y), \quad \frac{\partial S}{\partial y^0}(t, x, y) = -q_0(t, x, y)
\end{aligned}$$

Idea of proof: The partial derivatives are considered first. Since the inverses $\frac{\partial X}{\partial q_0}^{-1}$ and $\frac{\partial Y}{\partial p^0}^{-1}$ exist due to Corollary 3.1 the functions S and σ are identical up to the diffeomorphism given in Theorem 3.1. Now the derivatives follow by direct calculation. That means that from this point the proof follows the line of arguments for the nondegenerate variational principle, see e.g. [Kol00].

Proposition 3.4. *Let the assumptions of Proposition 3.1 be satisfied, then the following representations hold:*

$$\begin{aligned}
\frac{\partial^2 S}{\partial x^2} &= \frac{\partial P}{\partial p^0} \left(\frac{\partial X}{\partial p^0} \right)^{-1}, & \frac{\partial^2 S}{\partial y^2} &= \frac{\partial Q}{\partial q_0} \left(\frac{\partial Y}{\partial q_0} \right)^{-1}, \\
\frac{\partial^2 S}{\partial x \partial y} &= \frac{\partial Q}{\partial p^0} \left(\frac{\partial X}{\partial p^0} \right)^{-1}, & \frac{\partial^2 S}{\partial y \partial x} &= \frac{\partial P}{\partial q_0} \left(\frac{\partial Y}{\partial q_0} \right)^{-1},
\end{aligned}$$

where the functions S, X, P, Q *depend on the same tuple* (t, x, y, q_0, p^0) *of variables.*

This is achieved by iteration from Proposition 3.3 and Corollary 3.1 which implies Theorem 3.1.

Remark 3.1. *Due to the symmetry of the results in Corollary 3.1 and Proposition 3.3 an analogue to Proposition 3.4 in the variables x_0, y^0 holds. For the lowest order approximation the following explicit expressions exist for each component:*

$$\frac{\partial^2 S}{\partial^2 \{x,y\}} = \begin{pmatrix} \frac{12}{t^3}(1+\delta) & \frac{-6}{t^2}(1+\delta) \\ \frac{-6}{t^2}(1+\delta) & \frac{4}{t}(1+\delta) \end{pmatrix} \quad and \quad \frac{\partial^2 S}{\partial^2 \{x_0,y^0\}} = \begin{pmatrix} \frac{12}{t^3}(1+\delta) & \frac{6}{t^2}(1+\delta) \\ \frac{6}{t^2}(1+\delta) & \frac{4}{t}(1+\delta) \end{pmatrix}. \tag{24}$$

For the Hamiltonian $H(\xi,\eta)$ let us define the functional

$$I_t(\xi) = \int_0^t \max_\eta \left\{ (\eta,\dot{\xi}) - H(\xi,\eta) \right\}(\tau)d\tau \tag{25}$$

The maximum is taken over all piecewise smooth curves ξ with given endpoints $\xi(0) = \xi_0$ and $\xi(t) = \varkappa$.

Proposition 3.5. *For the Hamiltonian given in (2) the characteristic curve $(X,Y)(t,x_0,y^0,q_0,p^0)$ provides a minimum for the functional (25) over all curves in $D(x_0,y^0) = B_{\frac{c^2}{t^2}}(q_0) \times B_{\frac{c^3}{t^3}}(p^0)$. Furthermore, $S(t,x,y,x_0,y^0)$, of (23), constitutes the unique minimal value of the functional (25).*

Proof. For the Hamiltonian given in (2) the Weierstraß condition is satisfied, i.e. for all $t > 0$ and all ξ, $\eta \in \mathbb{R}^d$: $W(x(t),y(t),\xi,\eta,p(t),q(t)) \geq 0$ for arbitrary solutions $(x(t),y(t),q(t),p(t))$ of (9) where the Weierstraß function is given by

$$W(x,y,\xi,\eta,p,q) = \Delta H - L(H)$$

with H as defined in (2), $\Delta H := H(x,y,q_1,p_1) - H(x,y,q_2,p_2)$ and $L(H) = \left(\Delta q, \frac{\partial H}{\partial q}\right) + \left(\Delta p, \frac{\partial H}{\partial p}\right) := \left(q_1 - q_2, \frac{\partial H}{\partial q}\right) + \left(p_1 - p_2, \frac{\partial H}{\partial p}\right)$. Then we find

$$\Delta H = \frac{1}{2}(g(x)q_1,q_1) - (g(x)q_2,q_2) - (G(x)y,\Delta p) - \left(\frac{\partial}{\partial x}(G(x)y,y),\Delta q\right)$$

$$L(H) = (q_1 - q_2, g(x)q_2) + \frac{1}{2}\left(\frac{\partial}{\partial x}(G(x)y,y),\Delta q\right) - (G(x)y,\Delta p).$$

Combining the two equations reveals

$$W = \frac{1}{2}(g(x)q_1,q_1) - (q_1,g(x)q_2) + \frac{1}{2}(g(x)q_2,q_2) = \frac{1}{2}(q_1 - q_2, g(x)(q_1 - q_2)) + O(1) \geq 0,$$

the positivity holding since the matrix g is positive definite. As a consequence the Hamilton function treated in this paper satisfies the Weierstrass condition. Inserting definitions and Theorem 3.1 we have $I_t \geq S$ c.f. (23). Hence the minimum for the functional will be unique for arbitrarily chosen characteristics in $B_{\frac{c^2}{t^2}}(q_0) \times B_{\frac{c^3}{t^3}}(p^0)$, which finishes the proof of Proposition 3.5. □

Combining the result of Theorem 3.1 with the uniqueness result obtained in Proposition 3.5 gives us the equivalence of solutions to the Hamilton equation and Hamilton Jacobi equation.

We conclude this section by showing that the two-point function S is convex on $D(x_0, y^0)$. For arbitrary square matrices a, b, c, d the general following formula holds

$$det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = det \ d \quad det \ (a - bd^{-1}c) \ .$$

Using the asymptotics given in Remark 3.1 as a starting point, applying the above result and integrating twice w.r. to x and y we see that the two point function can be uniformly approximated by a quadratic function in x and y. In particular the two point function S can be approximated as follows

$$S(x, y, x_0, y_0) = \frac{6}{t^3}(1 + \delta)(x - x_0)^2 + \frac{6}{t^2}(1 + \delta)\left((x - x_0), y + y^0\right)$$
$$+ \frac{2}{t}(1 + \delta)[(y, y) + (y, y^0) + (y^0, y^0)]. \qquad (26)$$

This leads to a Gaussian approximation of the solution to (1).

4 Uniform Estimates for the Heat Kernel

In this section we shall construct an asymptotic expansion for the fundamental solution (Greens function) of the degenerate diffusion (1). To this end the well known WKB-type method is modified in such a way that the variables representing derivatives i time of order zero to three are being transformed in order to lift them to the same time scale. Hereby closed algebraic equations for the coefficients of the expansion are achieved, see also the construction in [Kol00] for the origin of our construction.

We shall discuss two types of asymptotics for the solution to (1) and (2), namely, small time asymptotics when $t \to 0$ and h is fixed ($h = 1$) and small diffusion asymptotics when t is fixed and $h \to 0$.

We look for the asymptotic fundamental solution to (4) using a WKB Ansatz

$$u^0(t, x, y, x_0, y^0, h) = C(h)\phi(t, x, y, x_0, y^0)\exp[-S(t, x, y, x_0, y^0)/h] \ , \qquad (27)$$

with real functions ϕ and S called the amplitude and the phase, respectively. Moreover, C is a function only depending on h alone while ϕ and S are independent of h having the arguments $x, y, x_0, y^0 \in \mathbb{R}^d$. This particular form of the asymptotic fundamental solution applies due to the positivity of the resolving diffusion operator. It is specific to linear second order pseudo differential equations describing Markov processes.

This asymptotic ansatz is also called exponential. Hence one should require that the solution u of (1) differs from u^0 by a multiplicative error:

$$u(t,x,y,x_0,y^0,h) = C(h)\phi(t,x,y,x_0,y^0)\exp[-S(t,x,y,x_0,y^0)/h](1+O(h)). \quad (28)$$

For (28) only the term with minimal entropy of the sum survives at each point. Therefore, for (5) the common superposition principle transforms to the idempotent superposition principle $(S_1,S_2) \mapsto min(S_1,S_2)$ at the level of actions. For a detailed discussion of this idempotent superposition principle and its applications see [KM94], [KM97]. Inserting u^0 (27) into (1) reveals

$$h\left(\frac{\partial\phi}{\partial t} - \frac{1}{h}\phi\frac{\partial S}{\partial t}\right) = \frac{h^2}{2}tr\ g(x)\left(\frac{\partial^2\phi}{\partial y^2} - \frac{\phi}{h}\frac{\partial^2 S}{\partial y^2}\right) + h\left(G(x)y, \frac{\partial\phi}{\partial x} - \frac{1}{h}\phi\frac{\partial S}{\partial x}\right) \quad (29)$$
$$+\frac{-h}{2}\left(\frac{\partial}{\partial x}(G(x)y,y), \frac{\partial\phi}{\partial y} - \frac{1}{h}\phi\frac{\partial S}{\partial y}\right) + \frac{1}{2}\left(g(x)\frac{\partial S}{\partial y}\right)\phi - h\left(g(x)\frac{\partial S}{\partial y}, \frac{\partial\phi}{\partial y}\right)$$

The terms in the above equation may be classified according to the degree of h which ranges from 0 to 2. Comparing coefficients we obtain the following two equations: The first Hamilton-Jacobi equation (h to the order 0):

$$\frac{\partial S}{\partial t} + \frac{1}{2}\left(g(x)\frac{\partial S}{\partial y}, \frac{\partial S}{\partial y}\right) - \left(G(x)y, \frac{\partial S}{\partial y}\right) + \frac{1}{2}\left(\frac{\partial}{\partial x}(G(x)y,y), \frac{\partial S}{\partial y}\right) = 0 \quad (30)$$

which in terms of the Hamiltonian function (2) gives

$$\frac{\partial}{\partial t}S + H\left(x,y,\frac{\partial S}{\partial x},\frac{\partial S}{\partial y}\right) = 0 \quad .$$

This way the system given by the PDE (4) is linked with the Hamilton-Jacobi equation studied in Section 2. The second equation is the so called transport equation (the coefficient of h^1):

$$\frac{\partial\phi}{\partial t} + \left(g(x)\frac{\partial S}{\partial y}, \frac{\partial\phi}{\partial y}\right) + \frac{1}{2}tr\left(g(x)\frac{\partial^2 S}{\partial y^2}\right) - \left(G(x)y, \frac{\partial\phi}{\partial x}\right) + \frac{1}{2}\left(\left(\frac{\partial G(x)}{\partial x}y,y\right),\frac{\partial\phi}{\partial y}\right) = 0.$$
$$(31)$$

Going the opposite way by choosing S and ϕ which satisfy (30) and (31) and inserting then into the r.h.s of (1) we define a function u^0 which satisfies equation (1) up to an error of order h^2. More precisely we obtain the following inhomogeneous equation

$$h\frac{\partial u^0}{\partial t} - H\left(x,y,-h\frac{\partial}{\partial x},-h\frac{\partial}{\partial y}\right)u^0 = -\frac{h^2}{2}C(h)F_0(t,x,y,x_0,y^0) \quad (32)$$

$F_0(t,x,y,x_0,y^0) \equiv tr\left(g(x)\frac{\partial^2\phi}{\partial y^2}\right)\exp(-\frac{S(t,x,y)}{h})$ i.e. the right hand side defines a function $F_0(t,x,y,x_0,y^0)$. If we would concentrate on a Cauchy problem instead with a

smooth initial function then we would have to solve an appropriate Cauchy problem for the Hamilton-Jacobi equation.

We now proceed to derive the precise asymptotic behaviour of the solution to the parabolic equation (1) for small times (respectively in a small neighbourhood of the starting point) we shall provide asymptotic expansions in the time parameter t for the two point function S and the amplitude ϕ given in (27). Studying the inhomogeneous system (32) has the advantage that the functions S and ϕ satisfy the Hamilton-Jacobi- (30) respectively the transport equation (31).

In Section 3 of this paper we have seen that the two point function S in 23 is smooth and satisfies the Hamilton-Jacobi equation almost everywhere. Moreover, it was shown that the solution of the Hamilton-Jacobi equation and the Hamilton equations are equivalent thus generalizing the results given in [Kol00] for non degenerate systems to a special case of a regular Hamiltonian of rank one. Due to the results in Section 3 we only use the expansions of the solution to the Hamiltonian system in the sequel.

In a fist step we neglect the inhomogeneity in (32). Starting with the phase respectively the two point function we are looking for a representation of the form

$$S(t,x,y) = \sum_{i=-k}^{\infty} S_i(x,y)t^i \quad . \tag{33}$$

Traditionally this is achieved by introducing the asymptotic expansion for the corresponding Hamiltonian system (9) into the Hamilton-Jacobi equation at the point (t,x,y) and subsequently solving a closed system of algebraic equations in x and $y \in \mathbb{R}^d$. This fails in the case of our degenerate system. In this paper we proceed by introducing a change of variables which first shifts the origin into the solution $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q})$ of the Hamilton equation (9) starting at $(x_0, y_0, 0, 0)$ and then adjusts the time scales of the variables on the manifold and the tangent space. We define a function $\sum$ on $\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d$ by

$$\sum(t,\xi,y) = S(t, t(x + \tilde{x}(t)), y + \tilde{y}, x_0, y^0) \ . \tag{34}$$

We remark that the function $\sigma(t,x,y) = S(t, x + \tilde{x}(t), y + \tilde{y}, x_0, y^0)$ coincides with the one given in (23). There holds $\frac{\partial \Sigma}{\partial \xi} = t\frac{\partial \sigma}{\partial x}$ and $\frac{\partial \Sigma}{\partial t} = \frac{\partial \sigma}{\partial t} + x\frac{\partial \sigma}{\partial x}$. In this way a closed system of algebraic equations for the coefficient functions S_i of an expansion of the phase function in powers of x and y will be derived.

The equation (30) becomes

$$0 = \frac{\partial \Sigma}{\partial t} - \frac{\xi + G(t\xi + \tilde{x})(y + \tilde{y}) - G(\tilde{x})\tilde{y}}{t}\frac{\partial \Sigma}{\partial \xi} - \left(g(\tilde{x})\tilde{q}, \frac{\partial \Sigma}{\partial y} \right) \tag{35}$$

$$+ \frac{1}{2}\left[\left(\frac{\partial G(t\xi + \tilde{x})}{\partial x}(y + \tilde{y}), (y + \tilde{y}) \right) - \left(\frac{\partial G(\tilde{x})}{\partial x}\tilde{y}, \tilde{y} \right) \right]\frac{\partial \Sigma}{\partial y} + \frac{1}{2}\left(g(t\xi + x)\frac{\partial \Sigma}{\partial y}, \frac{\partial \Sigma}{\partial y} \right)$$

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Proposition 4.1. *For the transformed two-point function Σ as defined in (34) there exists a unique asymptotic expansion in powers of $t > 0$ of the form*

$$\Sigma = \frac{\Sigma_{-1}}{t} + \Sigma_0 + t\Sigma_1 + t^2\Sigma_2 + ...$$

such that Σ_1 and Σ_0 vanish at the origin, Σ_1 is strictly convex in a neighbourhood of the origin and all Σ_j, $j \geq 2$, are regular power series in (ξ, y). Moreover, the remainder of arbitrary order can be estimated by means of induction using Proposition 3.2 as initialization.

Proof As in Section 3 let us replace the coefficient functions g, G in equation 35 by their n^{th} order Taylor series and let us also insert the n'^{th} order approximation of $(\tilde{x}, \tilde{y}, \tilde{p}, \tilde{q})$. Then the asymptotic result given in Corollary 3.2 gives that the combined error due to this replacement is of the order $0(t^{n+1})$.
This way (35) becomes in coordinate form (normal coordinates) for $n = 1$ up to an error of order $0(t^2)$:

$$\frac{\partial\Sigma}{\partial t} - \frac{(\xi + y)_i + \frac{t^2}{2}g_{ij}^{k\ell}[(\xi_k - y_k^0)(\xi_\ell - y_\ell^0)(y_j + y_j^0) - y_k^0 y_\ell^0 y_j^0] + O(t^3)}{t}\frac{\partial\Sigma}{\partial\xi_i}$$

$$- \frac{1}{2}[tg_{ij}^{k\ell}[(\xi_\ell - y_\ell^0)(y_i + y_i^0)(y_j + y_j^0) - y_i^0 y_j^0 y_\ell^0] + O(t^2)]\frac{\partial\Sigma}{\partial y_k} \qquad (36)$$

$$+ \frac{1}{2}(1 + \frac{t^2}{2}g_{ij}^{k\ell}(\xi_k - y_k^0)(\xi_{\ell k} - y_\ell^0) + O(t^3))\frac{\partial\Sigma}{\partial y_i}\frac{\partial\Sigma}{\partial y_j} = 0 \ .$$

We now rearrange the l.h. side of this equation according to the order in t. For the lowest orders in t we arrive at the subsequent set of equations. In particular for the lowest order in t, i.e. t^{-2}, equation (36) reduces to the following first order homogeneous partial differential equation with linear coefficient functions:

$$- \Sigma_{-1} - (y + \xi)\frac{\partial\Sigma_{-1}}{\partial\xi} + \frac{1}{2}(g_0\frac{\partial\Sigma_{-1}}{\partial y}, \frac{\partial\Sigma_{-1}}{\partial y}) = 0 \ . \qquad (37)$$

This linear first order partial differential equation is of the following general form

$$\lambda u + (Ax, \nabla u) = p(x) \qquad (38)$$

where $\lambda \in \mathbb{R}^+$, $x \in \mathbb{R}^m$, $p(x)$ is a homogeneous polynomial, and A is an $m \times m$ matrix with strictly positive eigenvalues $a_1 \leq a_2 \leq \ldots \leq a_m$. The solution constitutes a polynomial of a particular form given in the following lemma. The proof goes by inserting and direct calculations.

Lemma 4.1. *i) Let p be a homogeneous polynomial of degree q. Then the solution of (38) exists, moreover it constitutes a polynomial of degree q with coefficients of the form*

$$\frac{\partial^q u}{\partial x_{i_1} \ldots \partial x_{i_q}} = \frac{1}{a_{j_1} + \ldots + a_{j_q}} (C_{j_1 i_1}^{-1}) \cdots (C_{j_q i_q}^{-1}) \frac{\partial^q p}{\partial x_{\ell_1} \ldots \partial x_{\ell_q}} C_{j_1 i_1} \cdots C_{j_q i_q} .$$

Let $p = \sum_{q=0}^m p_q$ be a sum of homogeneous polynomials of degree q (in case $m = \infty$ the sum has to be absolutely convergent in a ball). Then the analytic solution of (38) exists and is given by the sum $\sum_{q=1}^m u_q$ of solutions u_q corresponding to the inhomogeneity p_q. In case $m = \infty$ the domains of convergence of p and the solution u to (38) coincide.

For a proof see [Kol00]. We now proceed to calculate the coefficient functions Σ_i, $-1 \le i$.

Lemma 4.2. *Under the additional assumption $\Sigma_{-1}(0,0) = 0$ and that Σ_{-1} is strictly convex at the origin the formal power series for Σ_{-1} in powers of ξ and y reduces to the quadratic form $6(\xi, \xi) + 6(\xi, y) + 2(y, y)$.*

Proof. Applying the assumption that Σ_{-1} has an asymptotic expansion

$$\Sigma_{-1} = \sum_{k \ge 0} \zeta_k$$

where ζ_k is a polynomial of order k in ξ and y and using the assumption $\Sigma_{-1}(0,0) = 0$ it immediately follows that $\zeta_0 = 0$. In local coordinates $g_0 = 1$. The first order term is of the form $\zeta_1 = K\xi + Ly$ where $K, L \in \mathbb{R}$. By inserting into (37) and using that $\zeta_0 = 0$ we get the equation

$$-K\xi - Ly - (y + \xi)K + \frac{1}{2}L^2 = 0 .$$

At the origin we find $L^2 = 0$. Differentiating with respect to y reveals $-K - L = 0$ and hence ζ_1 vanishes altogether, i.e $\zeta_1 \equiv 0$. Assuming for the quadratic part ζ_2 the representation

$$-\frac{1}{2}(A\xi, \xi) - (B\xi, y) + \frac{1}{2}(Cy, y) \tag{39}$$

we get the equations

$$A = \frac{1}{2}B^t B \quad 2B + A = CB \quad \frac{1}{2}C + B = \frac{1}{2}C^2.$$

Solving this algebraic system of equations we find, since (39) should be positive definite, $C = 4\ id, \quad B = 6\ id,$ and $\quad A = 12\ id$, i.e. (39) is of the form

$$6(\xi, \xi) + 6(\xi, y) + 2(y, y) \quad . \tag{40}$$

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Finally all ζ_k, $k \geq 3$, vanish, in particular for $k = 3$ the terms of order 3 in ξ, y make up the equation

$$0 = -\zeta_3 - \left((y+\xi), \frac{\partial \zeta_3}{\partial \xi} \right) + \left(-\frac{\partial \zeta_2}{\partial \xi}, -\frac{\partial \zeta_3}{\partial y} \right) = \zeta_3 + \left((y+\xi), \frac{\partial \zeta_3}{\partial \xi} - (6\xi + 4y), \frac{\partial \zeta_3}{\partial \xi} \right) . \tag{41}$$

where (40) has been inserted and the right hand side has been multiplied by -1. Due to Lemma 4.1 this implies that $\zeta_3 = 0$. Inserting ζ_k, $k \geq 3$, into (37) reproduces (41), hence ζ_k vanish f for $k \geq 3$, which finishes the proof of the lemma. $\qquad\square$

For the term Σ_0 corresponding to the order t^0 in the expansion of Σ we have the equation

$$-(y+\xi)\frac{\partial \Sigma_0}{\partial \xi} + (6\xi + 4y)\frac{\partial \Sigma_0}{\partial y} = 0 ,$$

the solution of which vanishes due to Lemma 4.1 since the equation is homogeneous. For the coefficients of t in (36) we obtain the equation

$$\Sigma_1 - (y+\xi)\frac{\partial \Sigma_1}{\partial \xi} + (6\xi + 4y)\frac{\partial \Sigma_1}{\partial y} = F_1(\xi, y) , \tag{42}$$

where $F_1(\xi, y)$ is the sum of the homogeneous polynomials F_{11}, F_{12} and F_{13} of order 2,3 and 4, respectively, in ξ and y. In particular we have

$$\begin{aligned}
F_{11} &= g_{ij}^{kl}[(12\xi_i\xi_k - 4y_iy_k)y_j^0 y_l^0 - (18\xi_iy_j + 7y_iy_j + 9\xi_i\xi_j)y_k^0 y_l^0 + (3\xi_k\xi_l + 2\xi_ky_l)y_i^0 y_j^0] \\
F_{12} &= g_{ij}^{kl}[(-6\xi_i\xi_k\xi_l + 3\xi_k\xi_l l y_i + 4\xi_k y_i y_l)y_j^0 \\
&\qquad + (36\xi_i\xi_k y_j + 11\xi_k y_i y_j - 2y_iy_jy_k + 18\xi_i\xi_j\xi_k)y_l^0] \\
F_{13} &= g_{ij}^{kl}[2\xi_k y_i y_j y_l - 4\xi_k\xi_l y_i y_j - 18\xi_i\xi_k\xi_l y_j - 9\xi_i\xi_j\xi_k\xi_l] .
\end{aligned}$$

Due to Lemma 4.1 the form of the solution Σ_1 is determined to be a polynomial of degree 4 in ξ, y and y^0. The expansion of Σ in powers of t (having expanded in x and y before) is explicitly given by

$$\begin{aligned}
\Sigma = \frac{6\xi_i{}^2 + 6\xi_i y_i + 2y_i{}^2}{t} + t \Big(& -\tfrac{14}{15}\, g_{i,j}^{k,l} y_{0,k} y_{0,l} y_i y_j - \tfrac{59}{20}\, g_{i,j}^{k,l} y_{0,k} y_{0,l} \xi_i y_j + \tfrac{3}{5}\, g_{i,j}^{k,l} y_{0,k} y_{0,l} \xi_i \xi_j \\
& - \tfrac{2}{15}\, g_{i,j}^{k,l} y_{0,j} y_{0,l} y_i y_k + \tfrac{7}{5}\, g_{i,j}^{k,l} y_{0,j} y_{0,l} \xi_i y_k + \tfrac{24}{5}\, g_{i,j}^{k,l} y_{0,j} y_{0,l} \xi_i \xi_k + \tfrac{1}{15}\, g_{i,j}^{k,l} y_{0,i} y_{0,j} y_l y_k \\
& + \tfrac{11}{20}\, g_{i,j}^{k,l} y_{0,i} y_{0,j} \xi_k y_l + \tfrac{3}{5}\, g_{i,j}^{k,l} y_{0,i} y_{0,j} \xi_k \xi_l + \tfrac{7}{10}\, g_{i,j}^{k,l} y_{0,l} \xi_i y_k y_j - \tfrac{9}{10}\, g_{i,j}^{k,l} y_{0,l} \xi_i \xi_j y_k \\
& - \tfrac{3}{2}\, g_{i,j}^{k,l} y_{0,l} \xi_i \xi_j \xi_k - \tfrac{1}{5}\, g_{i,j}^{k,l} y_{0,j} \xi_i y_l y_k - \tfrac{6}{5}\, g_{i,j}^{k,l} y_{0,j} \xi_i \xi_k y_l - 3\, g_{i,j}^{k,l} y_{0,j} \xi_i \xi_k \xi_l - \tfrac{3}{56}\, g_{i,j}^{k,l} \xi_i y_l y_k y_j \\
& + \tfrac{1}{7}\, g_{i,j}^{k,l} \xi_i \xi_j y_l y_k + \tfrac{5}{7}\, g_{i,j}^{k,l} \xi_i \xi_j \xi_k y_l + g_{i,j}^{k,l} \xi_i \xi_j \xi_k \xi_l \Big) + O(t^2) .
\end{aligned}$$

The calculations for higher orders go by analogy although they become substantially wilder. The terms Σ_j, $j > 1$, turn out to be homogeneous polynomials in ξ, y, y^0 of degree $j + 3$ which can be estimated analogously to the term Σ_1.

In normal coordinates in particular the term Σ_1 is given by $2.5 g_{ij}^{kl} y_i^0 y_j^0 y_k^0 y_l^0$ due to the symmetry of g_{ij}^{kl} which is why we omit it when transforming back. The inverse transformation $S(t, x, y, y^0) = \Sigma(t, \frac{x - \tilde{x}(t)}{t}, y - \tilde{y}(t))$ has been found explicitly for an approximation of order 1 using MAPLE, a programme for symbolic and numerical algebra, as being

$$S_i(t, x, y, x_0, y^0) = t^{-1} \left(-6 \frac{x_{0,i} y_i^0}{t} - 6 \frac{x_{0,i} y_i}{t} + 6 \frac{x_i y_i}{t} + 6 \frac{x_i y_i^0}{t} \right.$$
$$\left. - \quad 12 \frac{x_i x_{0,i}}{t^2} + 6 \frac{x_{0,i}^2}{t^2} + 6 \frac{x_i^2}{t^2} + 2 y_i^{0^2} + 2 y_i^2 + 2 y_i^0 y_i + O(t) \right) \ .$$

This is consistent with the estimate found in (24). On the diagonal this expression is of the form $S_i = 6 y_i^2 t^{-1}$ which obviously is not vanishing and hence reveals an exponential decay for the kernel on the diagonal.

For $x_0 = 0$ this reads:

$$S(t, x, y, y^0) = \frac{6}{t^3} [1 + O(|x|^2) + O(|y - y^0|^2)]|x|^2$$
$$+ \frac{6}{t^2} \langle x, y + y^0 + O(|x|^2) \rangle$$
$$+ \frac{2}{t} [|y|^2 + \langle y^0, y \rangle + |y^0|^2 + O(|x|^2) + O(|y - y^0|^2)] \ .$$

Having completed the first step in giving an approximate solution of exponential WKB type (27), namely, having derived an asymptotic expansion for the solution S to (30) we now turn to the transport equation (31). The parabolic equation (31) has the amplitude ϕ as the only unknown function left, for which we intend to give an asymptotic expansion in the sequel. As before we introduce a change of variables which adjusts the different time scales for the variables of the tangent bundle, in particular:

$$\psi(t, \xi, y) = t^\alpha \phi(t, t\xi + \tilde{x}(t), y + \tilde{y}(t)) \tag{43}$$

where the constant α is determined by consistency requirements. In terms of the transformed function the transport equation takes the form

$$0 = \frac{\partial \psi}{\partial t} - \frac{\alpha}{t} \psi + \frac{1}{2} \, tr \left(g(t\xi + \tilde{x}) \frac{\partial^2 \Sigma}{\partial y^2} \right) + (g(t\xi + \tilde{x}) \frac{\partial \Sigma}{\partial y}, \frac{\partial \psi}{\partial y}) - (g(\tilde{x})\tilde{q}, \frac{\partial \psi}{\partial y}) \tag{44}$$
$$-t^{-1}(\xi + G(t\xi + \tilde{x})(y + \tilde{y}) - G(\tilde{x})\tilde{y}, \frac{\partial \psi}{\partial \xi}) + \frac{1}{2}(\frac{\partial G}{\partial x}(t\xi + \tilde{x})(y + \tilde{y}), (y + \tilde{y}) \frac{\partial \psi}{\partial y})$$

Proposition 4.2. *There exists a unique $\alpha > 0$ such that there exists an asymptotic expansion for the solution of (28)*

$$\psi = \psi_0 + t\psi_1 + t^2\psi_2 + \ldots$$

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where each ψ_k, $k \geq 0$, constitutes a polynomial in (ξ, y) with degree strictly smaller than k and ψ_0 is some constant. Moreover, the solution is unique up to a constant multiplier.

Proof. Assuming the solution of (44) to be of the form of an asymptotic expansion (regular power series) the equation (44) decomposes into separate equations for each order in t, e.g. for the lowest order (t^{-1}) we find

$$-\alpha \psi_0 + \frac{1}{2} \psi_0 \; tr(g(x_0) \frac{\partial^2 \Sigma_{-1}}{\partial y^2}(x_0, y_0)) = 0 \;,$$

where we also have inserted the asymptotic expansion given by Proposition 4.1. This determines the parameter α in (43) to be

$$\alpha = \frac{1}{2} \; tr(g(x_0) \frac{\partial^2 \Sigma_{-1}}{\partial y^2}(x_0, y_0)) \quad . \tag{45}$$

Recalling that we are working in normal coordinates and reminding the representation given for Σ_{-1} in Lemma 4.2 we find $\alpha = 2n$. Comparing the coefficients of the term t^0 reveals the following equation

$$\psi_1 - \left(\xi + y, \frac{\partial \psi_1}{\partial \xi} \right) + (6\xi + 4y, \frac{\partial \psi_1}{\partial y}) = 0$$

the solution of which vanishes due to Lemma 4.1. In particular, this equation is the homogeneous counterpart to (32). For the coefficients of the term t^1 we get the following nonhomogeneous counterpart of the previous equation:

$$\psi_2 - \left(\xi + y, \frac{\partial \psi_2}{\partial \xi} \right) + \left(6\xi + 4y, \frac{\partial \psi_2}{\partial y} \right) + \; tr \left(\frac{1}{2} \frac{\partial^2 \Sigma_{-1}}{\partial y^2} + g^{kl}(\xi_k - y_k^0)(\xi_l - y_l^0) \right) = 0.$$

Decomposing the trace into its homogeneous polynomials of degree 0,1,2, respectively, we may determine the solution (a polynomial of degree 2 in (ξ, y, y^0)) as indicated by Lemma 4.1.

$$
\begin{aligned}
\psi = 1 + t^2 \Big(& - \left(-\tfrac{3}{56} \, g_{k,i}^{i,j} \xi_k - \tfrac{3}{56} \, g_{k,i}^{j,i} \xi_k - \tfrac{3}{56} \, g_{k,j}^{i,i} \xi_k \right) \xi_j + \tfrac{1}{6} \left(\tfrac{7}{10} \, g_{i,j}^{i,k} y_{0,j} - \tfrac{1}{5} \, g_{k,j}^{i,i} y_{0,j} \right. \\
& - \tfrac{3}{56} \, g_{k,i}^{i,j} y_j - \tfrac{3}{56} \, g_{k,i}^{j,i} y_j - \tfrac{3}{56} \, g_{k,j}^{i,i} y_j - g_{i,i}^{k,l} y_{0,l} \Big) y_k + \tfrac{5}{6} \left(\tfrac{7}{10} \, g_{i,j}^{i,k} y_{0,j} - \tfrac{1}{5} \, g_{k,j}^{i,i} y_{0,j} - \tfrac{3}{56} \, g_{k,i}^{i,j} y_j \right. \\
& - \tfrac{3}{56} \, g_{k,i}^{j,i} y_j - \tfrac{3}{56} \, g_{k,j}^{i,i} y_j - g_{i,i}^{k,l} y_{0,l} \Big) \xi_k - \tfrac{1}{6} \, g_{i,i}^{k,l} y_{0,k} y_l - \tfrac{5}{6} \, g_{i,i}^{k,l} y_{0,k} \xi_l + \tfrac{1}{210} \, g_{i,j}^{i,k} y_k y_j \\
& + \tfrac{3}{140} \, g_{i,j}^{i,k} \xi_j y_k + \tfrac{4}{35} \, g_{i,j}^{i,k} \xi_j \xi_k + \tfrac{1}{30} \, g_{i,i}^{k,l} y_l y_k + \tfrac{3}{20} \, g_{i,i}^{k,l} \xi_k y_l + \tfrac{4}{5} \, g_{i,i}^{k,l} \xi_k \xi_l \\
& - \tfrac{1}{30} \, g_{i,i}^{k,l} y_{0,k} y_{0,l} + \tfrac{1}{15} \, g_{i,j}^{i,k} y_{0,j} y_{0,k} - \tfrac{1}{30} \, g_{k,j}^{i,i} y_{0,k} y_{0,j} \Big) + O(t^3) \;.
\end{aligned}
$$

$$\tag{46}$$

$\square$

The inverse transformation $\phi(t, x, y) = \psi(t, \frac{x - \tilde{x}(t)}{t}, y - \tilde{y}(t))$ has been done explicitly for an approximation of order 1 using MAPLE. Applying this back transformation to ψ (which is given by an asymptotic expansion in t) will give back an asymptotic expansion in t. In particular, for $x_0 = 0$ we have

$$\phi(t, x, y, y_0) = (1 + O(|x| + |y - y^0| + t))t^{-\alpha} \quad .$$

Inserting the expansions for $S(t, x, y, x_0, y_0)$ and ϕ into the WKB Ansatz (27) reveals directly that the leading term is Gaussian.

We now proceed to show that the error term can be expressed by a factor. For an arbitrary m-dimensional system of ordinary differential equations $\frac{d}{dt}\mathcal{X}(t, \alpha) = f(\mathcal{X}, \alpha)$ depending on an m-dimensional parameter α a result of Liouville states that the square matrix $\mathcal{J} = [\frac{\partial \mathcal{X}}{\partial \alpha}]_{ij}$ is non degenerate on some time interval and satisfies $\frac{d}{dt}\mathcal{J} = \mathcal{J} \, tr[\frac{\partial f}{\partial \mathcal{X}}]$. For a proof see e.g. [FM81]. It is possible to determine the amplitude ϕ explicitly by applying this result to the system

$$\frac{d}{dt}x = \frac{\partial H}{\partial p}(x, y, p, q) \quad \text{and} \quad \frac{d}{dt}y = \frac{\partial H}{\partial q}(x, y, p, q)$$

where H is given in (2) and by using the equivalence of solutions for the Hamilton and the Hamilton-Jacobi equations given by Propositions 3.3 and 3.2. We find

$$p = \frac{\partial S}{\partial x}(t, x, y, x_0, y^0) \quad \text{and} \quad q = \frac{\partial S}{\partial y}(t, x, y, x_0, y^0)$$

to be dependent variables of x, y. Hence along the characteristics $(X, Y)(t, x_0, y^0, q^0, p_0)$ of the Hamilton equation corresponding to the Hamiltonian (2) the determinant $J = det\frac{\partial(X,Y)}{\partial(q^0, p_0)}$ satisfies the equation

$$\dot{J} = J \, tr\left(\begin{bmatrix} \frac{\partial^2 H}{\partial p \partial x} & \frac{\partial^2 H}{\partial p \partial y} \\ \frac{\partial^2 H}{\partial q \partial x} & \frac{\partial^2 H}{\partial q \partial y} \end{bmatrix} + \begin{bmatrix} \frac{\partial^2 H}{\partial p^2} & \frac{\partial^2 H}{\partial p \partial q} \\ \frac{\partial^2 H}{\partial q \partial p} & \frac{\partial^2 H}{\partial q^2} \end{bmatrix} \begin{bmatrix} \frac{\partial^2 S}{\partial x^2} & \frac{\partial^2 S}{\partial x \partial y} \\ \frac{\partial^2 S}{\partial y \partial x} & \frac{\partial^2 S}{\partial y^2} \end{bmatrix}\right)$$

where the second term in this sum corresponds to the dependent variables p, q. By inserting the definition of the Hamiltonian (2) most partial derivatives of H vanish leaving us with

$$\dot{J} = J tr\left(-\frac{\partial G}{\partial x}y + \frac{\partial G}{\partial x}y + \frac{\partial^2 H}{\partial q^2}\frac{\partial^2 S}{\partial y^2}\right) = J tr\left(g(x)\frac{\partial^2 S}{\partial y^2}\right) \tag{47}$$

which implies

$$\frac{d}{dt}J^{-\frac{1}{2}} = -\frac{1}{2}J^{-\frac{1}{2}} \, tr\left(g(x)\frac{\partial^2 S}{\partial y^2}\right) \quad . \tag{48}$$

On the other hand inserting the Hamilton equations into the transport equation (31) reveals

$$\frac{d}{dt}\phi + \frac{1}{2}\phi \ tr\left(\frac{\partial^2 H}{\partial q^2}\left(x, y, \frac{\partial S}{\partial x}, \frac{\partial S}{\partial y}\right)\frac{\partial^2 S}{\partial^2(x,y)}\right) = 0$$

or more explicitly

$$\frac{d}{dt}\phi + \frac{1}{2}\phi \ tr\left(g(x)\frac{\partial^2 S}{\partial y^2}\right) = 0 \quad . \tag{49}$$

Comparing (48) and (49) we find that $J^{-\frac{1}{2}}$ and ϕ are identical up to a constant factor α.

We now deal with the inhomogeneity in the PDE (32). Aiming at a multiplicative error for an exponential approximation of WKB type respectively a multiplicative error for u^0 we need uniform estimates for the inhomogeneity in an appropriate neighbourhood of the starting point (x_0, y^0). This will be achieved by exchanging ϕ with $J^{-\frac{1}{2}}$ and using the estimates given in Corollary 3.1 and (12).

Theorem 4.1. *Let us assume that the assumptions of Theorem 3.1 are satisfied in the whole cotangent bundle. Let an approximating solution of the form (28) to (1) satisfy the Hamilton Jacobi- (30) and transport equation (31). Then the inhomogenity F_0 (32) is a multiplicative error of the type*

$$\left[h\frac{\partial}{\partial t} - H\left(x, y, -h\frac{\partial}{\partial x}, -h\frac{\partial}{\partial y}\right)\right]u^0 = O(h^2)t^2u^0$$

Proof. Recalling the definition of F_0 and the fact that ϕ and $J^{-\frac{1}{2}}$ are identical up to a constant factor c.f. (48) and (49) it suffices to examine

$$\frac{\partial^\nu}{(\partial x_i)^\nu}J^{-\frac{1}{2}}(t, x, y, x_0, y^0) \quad \text{and} \quad \frac{\partial^\nu}{(\partial y_i)^\nu}J^{-\frac{1}{2}}(t, x, y, x_0, y^0)$$

for $\nu = 1, \ 2$ and $1 \leq i \leq n$. Let $p_0 = p_0(t, x, y, x_0, y^0)$ and $q_0 = q_0(t, x, y, x_0, y^0)$ satisfy the Hamilton Jacobi equation (30). Then we have along the characteristics $(X, Y)(t, x_0, y^0, q^0, p_0)$ of the Hamiltonian given in (2) for $\gamma = (\gamma_0, \gamma_1)$ with $\gamma_0 := x$, $\gamma_1 := y$ both in $\mathbb{R}^d$ and $\zeta = (q^0, p_0) \in \mathbb{R}^{2d}$

$$\begin{aligned}
\frac{\partial}{\partial\gamma_i}J^{-\frac{1}{2}} &= -\frac{1}{2}J^{-\frac{1}{2}}\left(J^{-1}\frac{\partial J}{\partial\gamma_i}\right) = -\frac{1}{2}J^{-\frac{1}{2}} \ tr\left[\left(\frac{\partial(X,Y)}{\partial\zeta}\right)^{-1}\frac{\partial}{\partial\gamma_i}\frac{\partial(X,Y)}{\partial\zeta}\right] \\
&= -\frac{1}{2}J^{-\frac{1}{2}}\left(\frac{\partial^2(X,Y)}{\partial\zeta\partial\zeta}\right)^k_{lm}\left(\frac{\partial(X,Y)}{\partial\gamma}\right)^{-1}_{li}\left(\frac{\partial(X,Y)}{\partial\zeta}\right)^{-1}_{mk}
\end{aligned} \tag{50}$$

since $\frac{\partial(X,Y)}{\partial\gamma_i} = \frac{\partial(X,Y)}{\partial\zeta}\frac{\partial\zeta}{\partial\gamma_i}$. Analogously, by iteration

$$\frac{\partial^2}{\partial\gamma_i^2}J^{-\frac{1}{2}} = -\frac{1}{2}\frac{\partial}{\partial\gamma_i}\left(J^{-\frac{3}{2}}\frac{\partial}{\partial\gamma_i}J\right) = -\frac{1}{2}\left[-\frac{3}{2}J^{-\frac{5}{2}}(\frac{\partial}{\partial\gamma_i}J)^2 + J^{-\frac{3}{2}}\frac{\partial^2}{\partial\gamma_i^2}J\right] = -\frac{1}{2}J^{-\frac{1}{2}}K$$

where

$$
\begin{aligned}
K : \quad = \quad & -\frac{3}{2}J^{-2}\frac{\partial}{\partial\gamma_i}J)^2 + J^{-1}\frac{\partial^2 J}{\partial\zeta^l\partial\zeta^n}\frac{\partial\zeta^l\partial\zeta^n}{\partial\gamma_i^2} + J^{-1}\frac{\partial}{\partial\zeta^l}J\frac{\partial^2\zeta^l}{\partial\gamma_i^2} \\
= \quad & -\frac{3}{2}\left(\frac{\partial(X,Y)}{\partial\zeta}\right)_{mk}^{-2}\left(\left(\frac{\partial^2(X,Y)}{\partial\zeta^2}\right)_{lm}^{k}\left(\frac{\partial(X,Y)}{\partial\gamma}\right)_{li}^{-1}\right)^2 \\
& + \left(\frac{\partial(X,Y)}{\partial\zeta}\right)_{mk}^{-1}\left(\frac{\partial^3(X,Y)}{\partial\zeta^l\partial\zeta^m\partial\zeta^n}\right)\left(\frac{\partial(X,Y)}{\partial\gamma}\right)_{li}^{-1}\left(\frac{\partial(X,Y)}{\partial\zeta}\right)_{ni}^{-1} \\
& + \left(\frac{\partial(X,Y)}{\partial\zeta}\right)_{km}^{-1}\left(\frac{\partial^2(X,Y)}{\partial\zeta^l\partial\zeta^m}\right)^{k}\left(\frac{\partial^2\zeta}{\partial\gamma_i^2}\right)^{lm} .
\end{aligned}
$$

Here the last factor may be rewritten as follows $\frac{\partial^2\zeta}{\partial\gamma_i^2} = -\left(\frac{\partial\gamma}{\partial\zeta}\right)_{\ell i}^{-2}\frac{\partial^2\gamma_i}{\partial\zeta^l\partial\zeta^n}\left(\frac{\partial\gamma}{\partial\zeta}\right)_{in}^{-1}$. In order to simplify notations we replace I_m by m, etc. in the next formula where I_m was defined in (19). Applying Corrollary 3.1 we find the following asymptotic behaviour for K:

$$
\begin{aligned}
& O(t^2) + O(t^{-(3-m-k)})O(t^{3\cdot3-k-\ell-m-n})O(t^{-(3-\ell-i)})O(t^{-(3-n-i)}) + O(t^{-(3-k-m)}) \\
& \times \quad O(t^{3\cdot3-k-\ell-m-n}t^{-(3-n-i)}t^{-(3-\ell-i)}) - O(t^{-2(3-\ell-i)}t^{2\cdot3-i-n-\ell}t^{2\cdot3-m-k-\ell}t^{-(3-i-n)}) \\
& = \quad O(t^2 i) + O(t^2 i) + O(t^2 i) .
\end{aligned}
$$

Recalling that $i = I_i = 1$ for y_i, $1 \le i \le d$, and inserting into (50) reveals for the leading terms

$$
\frac{\partial}{\partial y_i}J^{-\frac{1}{2}} = J^{-\frac{1}{2}}O(t^1) \tag{51}
$$

and by iteration

$$
\frac{\partial^2}{\partial y_i^2}J^{-\frac{1}{2}} = J^{-\frac{1}{2}}O(t^2) \tag{52}
$$

which finishes the proof of Theorem 4.1. $\qquad\square$

Next we relax the assumption that Theorem 4.1 hold in the whole cotangent bundle by introducing a cutoff function in the definition of the function u_0. Let χ_D be a smooth function with values in $[0, 1]$ such that χ_D vanishes outside of the domain D of the cotangent bundle, is 1 for all points (x, y) in the domain D except in a neighbourhood of the boundary ∂D of D, moreover, all partial derivatives of χ_D until order one in x and until order two in y are bounded for $t \in (0, t_0)$, $t_0 > 0$. Then the function

$$
u^D = C(h)\chi_D(x - x_0, y - y_0)\phi(t, x, y, x_0, y_0)\exp[-S(t, x, y, x_0, y_0)/h] \tag{53}
$$

is globally well-defined for $t \leq t_0$ and satisfies

$$h\frac{\partial u^D}{\partial t} - \frac{h^2}{2}\, tr\left(g(x)\frac{\partial^2 u^D}{\partial y_i \partial y_j}\right) - h\left(G(x)y, \frac{\partial u^D}{\partial x}\right) + \frac{h}{2}\left(\frac{\partial}{\partial x}(G(x)y, y), \frac{\partial u^D}{\partial y}\right)$$
$$= -h^2\, C(h)\, F(t, x, y, x_0, y^0) \tag{54}$$

where the inhomogeneity is now given by

$$
\begin{aligned}
F \;=\;& tr\left(G(x)\frac{\partial^2 \phi}{\partial y_i \partial y_j}\right)\chi_D \exp\left[-\frac{S}{h}\right] + \frac{h^{-1}}{2}\left(tr\left(g(x)\frac{\partial \chi_D}{\partial y_j}\frac{\partial S}{\partial y_i}\right)\phi\right.\\
&+ tr\left(g(x)\frac{\partial \chi_D}{\partial y_j}\frac{\partial \phi}{\partial y_i}\right) + \left.tr\left(g(x)\frac{\partial^2 \chi_D}{\partial y_j \partial y_i}\right)\phi\right)\exp\left[-\frac{S}{h}\right]\\
&+ h^{-2}\left(\left(G(x)y, \frac{\partial \chi_D}{\partial x}\right) - \frac{1}{2}\left(\frac{\partial}{\partial x}(G(x)y, y)\frac{\partial \chi_D}{\partial y}\right)\right)\phi\exp\left[-\frac{S}{h}\right].
\end{aligned}
$$

In the sequel we are going to study the order of magnitude in t and h of the inhomogeneity F. First we collect some general assumptions and facts. The appropriate neighbourhood is the polydisk $D = B_r(x_0) \times B_{\frac{r}{t}}(y_0)$ in particular its closure $\bar{D}$. The functions χ_D, $\frac{\partial}{\partial x}\chi_D$, $\frac{\partial}{\partial y}\chi_D$ and $\frac{\partial^2}{\partial y^2}\chi_D$ are bounded on D, moreover, g and $G = g^{-1}$ are bounded since the manifold is compact. In normal coordinates (actually $x_0 = 0$) $g(x) = id + G_{ij}x_i x_j + \ldots$ and $G(x) = id - G_{ij}(x - x_0)_i(x - x_0)_j + \ldots$ which implies that $O(\frac{\partial}{\partial x}g) = O(x - x_0) = O(-y_0 t) = c \wedge r$ and $O(\frac{\partial}{\partial x}G) = O(-(x - x_0)) = O(y_0 t) = c \wedge r$. Using the expansion given for S in (26) we find

$$
\begin{aligned}
\frac{\partial S}{\partial y_i} \;&=\; \frac{6}{t^2}(1-\delta)(x-x_0)_i + \frac{2}{t}(1-\delta)(y-y_0)_i\\
&=\; \frac{6}{t^2}(1-\delta)c \wedge r + \frac{2}{t}(1-\delta)(y_0 + tq_0 + y_0 + O(t))_i.
\end{aligned}
$$

this implies $O(\frac{\partial S}{\partial y_i}) = O(t^{-2}) + O(t^{-2}) = O(t^{-2})$. Using Proposition 3.3 we get similarly $O(\frac{\partial S}{\partial y_i}) = O(q_0) = O(t^{-2})$. Applying (50) to $\frac{\partial \phi}{\partial y_i}$ and collecting terms Theorem 4.1 is modified by the cutoff χ_D showing the following order of magnitude in t and h for the inhomogeneity F:

$$
\begin{aligned}
O(t^2)u^D \;&+\; \left(\frac{h^{-1}}{2}(O(t^{-2}) + O(t) + 1) + h^{-2}(O(t^{-1}) + O(t^1)O(t^{-2})\right)\chi_{\Delta D_\varepsilon}\phi\exp\left[-\frac{S}{h}\right]\\
&\leq\; O(t^2)u^D + \frac{h^{-1}}{2}(O(t^{-2}) + 1 + O(t))\chi_{\Delta D_\varepsilon}O\left(e^{-\frac{\Omega_0 - \delta}{ht - 3}}\right).
\end{aligned}
$$

where $\chi_{\Delta D_\varepsilon}$ stands for $\frac{\partial}{\partial y}\chi_D$ as well as $\frac{\partial^2}{\partial y_i y_j}\chi_D$. The support ΔD_ε of these functions is a neighbourhood of the boundary ∂D of width ε. In the above estimate we used that for given $\delta > 0$, $h > 0$ one can choose t_0 such that for $t \leq t_0$ and $h \leq h_0$

$$12^n (2\pi t)^{-2n} \leq \exp\left[\frac{\delta}{ht^3}\right], \qquad \frac{1}{ht^3} \leq \exp\left[\frac{\delta}{ht^3}\right].$$

24

The constant $\Omega = \Omega_0 + \delta$ summarizes the constants $\Omega_0 = \min\{t^3 S(z,\xi)| \, |x - x_0| = r - \varepsilon$ and $|y - y_0| = \frac{r - \varepsilon}{t}\}$ and δ. These calculations are summarized in the following proposition.

Proposition 4.3. *Under the assumptions of Theorem 3.1 the inhomogeneity F has the asymptotic behaviour $O(t^2)u^D + O(\exp[-\frac{\Omega}{ht^{-3}}])\chi_{\Delta D_\varepsilon}$ for $t, h > 0$ and some positive constant Ω.*

After having studied the solution (53) of WKB shape we passed on to solving the inhomogeneous equation (54). We now reconstruct the solution to the original homogeneous equation using the well known DuHamel principle. Recalling the classical results of Egorov, Hörmander, Oleinik and others we know that equation (4) is hypoelliptic since the generator $\mathcal{L}$ is of the form

$$\mathcal{L} = \Sigma_{i=1}^n X_i^2 + X_0$$

where $X_i = A_{ij}\frac{\partial}{\partial y_j}$, $1 \leq i, j \leq n$ with $A^*A = G$, $det A \neq 0$, and $X_0 = (G(x)y, \frac{\partial}{\partial x}) - \frac{1}{2}\left(\frac{\partial}{\partial x}(G(x)y, y), \frac{\partial}{\partial y}\right)$, moreover, the Lie-brackets $\left[X_i, (G(x)y, \frac{\partial}{\partial x})\right] = (A^*)_{i\ell}^{-1}\frac{\partial}{\partial x}$, $1 \leq i \leq n$, together with the vector fields X_i, $1 \leq i \leq n$, form a basis of the tangent space. In particular it is shown cf. Hörmander [Hör68], Oleinik [Ole71] that for any initial distribution in $\mathcal{D}'$ and smooth inhomogeneity there exists a unique smooth solution to (4). From this it immediately follows that the operators $\left(e^{t\mathcal{L}}\right)_{t>0}$ possess a smoothing kernel. For the sake of a shorter notation we are using $z = (x, y)$ and $z_0 = (x_0, y^0)$.

Proposition 4.4. *For any domain D in $\mathbb{R}^{2d}$ and C^2 function f with support in D there exists a unique smooth solution v_t in D satisfying the inhomogeneous equation*

$$\frac{\partial}{\partial t}v - \mathcal{L}v = f \ .$$

In fact, v is given by the integral formula

$$v(t, z, z_0) = u(t, z, z_0) - \int_0^t \int_{\mathbb{R}^{2n}} u(t - \tau, z, \xi)f(\tau, \xi, z_0) \, d\xi d\tau$$

where u satisfies the corresponding homogeneous equation (1).

Proof. Uniqueness follows from the uniqueness of the corresponding homogeneous equation (4). As for existence, applying the operator $\frac{\partial}{\partial t} - \mathcal{L}$ to the r.h.s. above yields

$$\left(\frac{\partial}{\partial t} - \mathcal{L}\right)v = \left(\frac{\partial}{\partial t} - \mathcal{L}\right)u(t, z, z_0) - \left(\frac{\partial}{\partial t} - \mathcal{L}\right)\int_0^t \int_{\mathbb{R}^{2n}} u(t - \tau, z, \xi)f(\tau, \xi, z_0) \, d\xi d\tau$$

$$
\begin{aligned}
= \quad & -\frac{\partial}{\partial t} \int_0^t \int_{\mathbb{R}^{2n}} u(t-\tau,z,\xi) f(\tau,\xi,z_0) \, d\xi d\tau + \mathcal{L} \int_0^t \int_{\mathbb{R}^{2n}} u(t-\tau,z,\xi) f(\tau,\xi,z_0) \, d\xi d\tau \\
= \quad & \int_{\mathbb{R}^{2n}} u(0,z,\xi) f(\tau,\xi,z_0) \, d\xi - \int_0^t \int_{\mathbb{R}^{2n}} \frac{\partial}{\partial t} u(t-\tau,z,\xi) f(\tau,\xi,z_0) \, d\xi d\tau \\
& + \mathcal{L} \int_0^t \int_{\mathbb{R}^{2n}} u(t-\tau,z,\xi) f(t,\xi,z_0) \, d\xi d\tau \\
= \quad & f(t,z_0,z_0) \ .
\end{aligned}
$$

We have used that u solves the homogeneous equation to eliminate terms. Furthermore when differentiating the parameter integral w.r. to t we have skipped the regularization needed for the argument giving $\int_{\mathbb{R}^{2n}} \delta_z(\xi) f(t,\xi,z_0) \, d\xi$ and have exploited the cutoff given by the function f for obtaining the integrability of $(\frac{\partial}{\partial t} u) f$. $\qquad \square$

For the homogeneous equation (1) and its nonhomogeneous counterpart (54) the solutions are therefore related in the following way

$$
u^D = (1 - hI_t)u \ ,
$$

with u^D given by (53) and where the operator I_t is defined by $I_t u(t,z,z_0) := \int_0^t \int_{\mathbb{R}^{2n}} u(t-\tau,z,\xi) F(\tau,\xi,z_0) \, d\xi d\tau$ with F given in (54). In case the inverse operator exists, which is equivalent to the convergence of its associated formal series $I + \sum h^m I_t^m$, we retrieve the solution to (1).

Theorem 4.2. *For small t the fundamental solution u to (1) has the form*

$$
u = u^D(1 + O(ht)) + O(\exp[-\frac{\Omega}{ht^{-3}}])
$$

with u^D given by (53) and Ω as in Proposition 4.3.

In order to meet the assumptions of Proposition 3.1, Theorem 3.1, and Proposition 4.3 we choose $D = B_r(x_0) \times B_{\frac{r}{t}}(y_0)$ for r sufficiently small. Due to Proposition 4.3 the action functional S satisfies $\min_\xi \{S(t-\tau,z,\xi) + S(\tau,\xi,z_0)\} = S(t,z,z_0)$. Introducing a zero term, using that S is a convex function on D with a unique minimum, and replacing $S(t-\tau,z,\xi) + S(\tau,\xi,z_0) - \min_\xi \{S(t-\tau,z,\xi) + S(\tau,\xi,z_0)\}$ by its Taylor polynomial of order three at ξ_0 minimizing $S(t-\tau,z,\xi) + S(\tau,\xi,z_0)$ we find:

$$
\begin{aligned}
I_t u^D(z) \leq \quad & C \int_0^t \int_{\mathbb{R}^{2d}} u^D(t-\tau,z,\xi) \left(\tau^2 u^D(\tau,\xi,z_0) + e^{-\frac{\Omega}{h\tau^3}} \chi_{\Delta D_\varepsilon}(\xi - z_0) \right) d\xi d\tau \\
\leq \quad & e^{-\frac{1}{h}S(t,z,z_0)} \int_0^t \frac{C(h)^2 \tau^2}{(t-\tau)^{2d}\tau^{2d}} \int_{\mathbb{R}^{2d}} e^{-\frac{1}{h}((\Xi_{f,t-\tau}+\Xi_{b,\tau})(\xi-\xi_0),\xi-\xi_0)} \chi_D(\xi - z_0) \chi_D(z - \xi) \, d\xi \, d\tau \\
+ \quad & C(h) \int_0^t \int_{\mathbb{R}^{2d}} (t-\tau)^{-2d} e^{-\frac{1}{h}S(t-\tau,z,\xi)} \chi_D(z-\xi) e^{-\frac{\Omega}{h\tau^3}} \chi_{\Delta D_\varepsilon}(\xi - z_0) \, d\xi \, d\tau
\end{aligned}
$$

$$\leq \quad CC(h)t^{-2d}e^{-\frac{1}{h}S(t,z,z_0)}t\chi_{D\star^2}(z-z_0)$$

$$+C(h)\int_0^t e^{-\frac{\Omega}{h\tau^3}}e^{-M}\int_{\mathbb{R}^{2d}}(t-\tau)^{-2d}e^{-\frac{1}{h}(\Xi_{b,\tau}(z-\xi),z-\xi)}\chi_D(z-\xi)\chi_{D\star^2}(z-z_0)\,d\xi\,d\tau$$

$$\leq \quad C\left(C(h)^2 e^{-\frac{1}{h}S(t,z,z_0)}t^{-2d+1}+te^{-\frac{\Omega}{ht^3}}\right)\chi_{D\star^2}(z-z_0) \tag{55}$$

where $C(h)=\frac{12^d}{(4\pi ht)^{2d}}$ and $\chi_{D\star^2}(z-z_0)$ is the characteristic function of the poly disc $D^{\star 2}=\{(x,y)\in\mathbb{R}^{2d}|\,|x-x_0|\leq 2r,\,|y-y_0|\leq\frac{2r}{t}\}$. Moreover, above we have introduced the matrices: $\Xi_{b,t}^{ij}=\frac{\partial^2 S(t,z,\xi)}{\partial y_i'\partial y_j'}$, and $\Xi_{f,t}^{ij}=\frac{\partial^2 S(t,\xi,z_0)}{\partial y_i'\partial y_j'}$, $i\leq i,j\leq d$, having non vanishing 2×2-matrices on the diagonal as given in (24), where $\xi=(x',y')\in\mathbb{R}^{2d}$. The determinants of both matrices are given by $\frac{12^d}{\sigma^{2d}}$ with $\sigma\in\{t-\tau,\tau\}$. Their sum equals $t^{2d}\tau^{-2d}(t-\tau)^{-2d}$. All constants are included in the universal constant C. Finally Ω is given in Proposition 4.3. We have used $\chi_D(\xi-z_0)\chi_D(z-\xi)\leq\chi_{D\star^2}(z-z_0)$ and that the function $\exp(-\frac{\Omega}{ht^3})$ is isotone having the image $[0,1)$. The mth order term I_t^m follows by iteration. For the sake of a clear representation we introduce the following notations:

$$f(t,z,z_0) \quad = \quad t^{-2d}\chi_D(z-z_0)C(h)exp(-S(t,z,z_0)/h),$$

$$g_m(t,z,z_0) \quad = \quad \chi_{D\star^m}(z-z_0)\exp\left[-\frac{\Omega}{ht^3}\right]$$

with $z=(x,y)\in\mathbb{R}^{2d}$, $z_0=(x_0,y_0)\in\mathbb{R}^{2d}$ and $\chi_{D\star^m}(z-z_0)$ is the characteristic function of the polydisc $D^{\star m}=\{(x,y)\in\mathbb{R}^{2d}|\,|x-x_0|\leq mr,\,|y-y_0|\leq\frac{mr}{t}\}$. In terms of the new notation inequality (55) reads $f*f(t,z,z_0)\leq CC(h)f(t,z,z_0)$ together with $f*Cg_1(t,z,z_0)\leq Ctg_2(t,z,z_0)$ for some constant C. As to the higher order convolutions we apply:

$$|f*g_m(t,z,z_0)|=\int_0^t\int_{\mathbb{R}^{2d}}\chi_{D\star^m}(z-\xi)e^{-\frac{\Omega}{h(t-\tau)^3}}\frac{C(h)}{\tau^{2d}}\left|e^{-\frac{1}{h}S(\tau,\xi,z_0)}\right|\chi_D(\xi-z_0)\,d\xi\,d\tau$$

$$\leq \quad \chi_{D\star^{m+1}}(z-z_0)\int_0^t e^{-\frac{\Omega}{h(t-\tau)^3}}e^{-M}\int_{\mathbb{R}^{2d}}\frac{C(h)}{\tau^{2d}}e^{(\Xi_{f,t}(\xi-z_0),\xi-z_0)}\chi_D(\xi-z_0)\,d\xi\,d\tau$$

$$\leq \quad g_{m+1}(t,z,z_0)e^{-M}t.$$

where we have used that S is locally convex satisfying $S(t,z,z_0)\geq M+(\Xi_{f,t}(\xi-z_0),\xi-z_0)$ on D. Moreover, there holds

$$g_1*g_m(t,z,z_0)=\int_0^t\int_{\mathbb{R}^{2d}}\chi_{D\star^m}(z-\xi)e^{-\frac{\Omega}{h(t-\tau)^3}}e^{-\frac{\Omega}{h\tau^3}}\chi_D(\xi-z_0)\,d\xi\,d\tau$$

$$\leq \quad \chi_{D\star^{m+1}}(z-z_0)e^{-\frac{\Omega}{ht^3}}\int_0^t e^{-\frac{\Omega}{h\tau^3}}\int_{\mathbb{R}^{2d}}\chi_D(\xi-z_0)\,d\xi\,d\tau$$

$$\leq \quad \pi^n\Gamma\left(\frac{d}{2}+1\right)^2 r^d\chi_{D\star^{m+1}}(z-z_0)e^{-\frac{\Omega}{ht^3}}\int_0^t\left(\frac{r}{t}\right)^d e^{-\frac{\Omega}{h\tau^3}}d\tau\leq tb(r)g_{m+1}(t,z,z_0).$$

where we have estimated $t^{-n}e^{-\frac{\Omega}{h\tau^3}} \le t^{-n}e^{-\frac{\Omega}{ht^3}} \le e^{-\frac{\Omega-\delta}{ht^3}} \le 1$ and $b(r)$ summarizes the constants. Hence by induction for the mth order term we find

$$
\begin{aligned}
h^{m-1}I_t^{\otimes m-1}u^D(z) &\le CC(h)(Chb(r)t)^{m-1}(f + 2g_2 + \ldots 2^{m-1}g_m) \\
&\le CC(h)(Chb(r)t)^{m-1}(f + g_m).
\end{aligned}
$$

This is applied when studying convergence, i.e. estimating $\left|(\sum_0^\infty h^m I_t^{\otimes m})u^D - u\right|(t,x,y,x_0,y^0)$. Since $Chb(r)t < 1$ for $h < h_0$ and $t < t_0$, h_0, t_0 sufficiently small, we get

$$
C(h)2C\sum_{\ell=m}^\infty (Chb(r)t)^\ell g_\ell \le C(h)\frac{(2Chb(r)t)^m}{1-(2Ch_0b(r)t_0)^m}e^{-\frac{\Omega}{ht^3}} \le \frac{(2Chb(r)t)^m}{1-(2Ch_0b(r)t_0)^m}e^{-\frac{\Omega-\delta}{ht^3}}
$$

where the ratio tends to zero for $m \to \infty$. Moreover, we also get that $C(h)f\sum_{\ell=m}^\infty (Chb(r)t)^\ell \le C(h)f\frac{(2Chb(r)t)^m}{1-(2Ch_0b(r)t_0)^m}$ tends to zero for $m \to \infty$. Hence the series $\sum_0^\infty h^m I_t^{\otimes m}$ converges uniformly. Hence we have shown that the solution u of the homogeneous system is given by a convergent series on any compact set if h is small.

Remark The result is local in as much as u^D is localized in D. But the Ω in the above estimates in Theorem 4.2 is uniform. Since the manifold is compact this result is enough for getting global results. This concludes the studies of the heat kernel. Our result is related to the construction of a generalized heat kernel on a bundle over a compact manifold in [BGV91] concedrning the structure of the underlying space, however the diffusion is degenerate.

5   A trace formula

In this last section we use the uniform bounds derived above to derive a trace formula, i.e. an asymptotic expansion for the trace of the semigroup describing the evolution of the solution to equation (1). Let us explain the most important facts. Let H and H' be (separable, infinite dimensional) Hilbert spaces, and choose orthonormal bases $(e_i)_{i\in\mathbb{N}}$ and $(e_j)_{j\in\mathbb{N}}$ in H and H', respectively. For any bounded operator $A : H \to H'$ the quantity $\|A\|_{HS} = \sum_{i,j}\langle Ae_i, e_j\rangle$ is independent of the choice of bases in H and H', c.f. [Sim79][Roe98]. An operator if it is finite such that $\|A\|_{HS} < \infty$ is called an *Hilbert Schmidt operator* and $\|A\|_{HS}$ is called its *Hilbert Schmidt norm*. The Hilbert Schmidt norm is induced by a scalar product $\langle A, B\rangle = \sum_{i,j}\langle Ae_i, e_j\rangle\langle Be_j, e_i\rangle$. A bounded operator is of *trace class* if there are Hilbert Schmidt operators A and B on H with $T = AB$. Its *trace* $Tr(T)$ is defined to be the Hilbert Schmidt inner product $\langle A^*, B\rangle_{HS}$. The trace $Tr(T)$ depends in fact only on T i.e. it is independent of the choice of A and B.

In [BGV91] chapter 2 a covariant approach to a generalized Laplacian with corresponding heat kernel on bundles over a compact orientable Riemannian manifold M is developed. In particular, the scalar product between the space of compactly supported sections of the bundle $\mathcal{E}$ and the space of sections of $\mathcal{E}^* \otimes |\Lambda|$ is established, where $|\Lambda|$ is the density bundle. The result given in Theorem 12.60 in [CFKS87] can be adopted. Since the generator $\mathcal{L}$ given by (1) considered in this paper is nonsymmetric we will, however, mainly exploit the existence of a kernel instead of working with the operators. Since the kernel is smoothing on the space of distributions according to [Hör68] and [Ole71] and since it constitutes the transistion probability of a diffusion the operators $\exp[t\mathcal{L}]$, $t > 0$, the corresponding Hilbert Schmidt norm is given by

$$
\begin{aligned}
\left\| e^{t\mathcal{L}} \right\|_{HS} &= \int_{T^*M} u(x,y,x,y)\,dxdy \leq \int_M \int_{TM} u^D(x,y,x,y)(1+O(t))dy\,dx \\
&\leq \mathrm{vol}(M) \int_{TM} e^{\frac{1}{\hbar}6y^2}(1+O(t))dy < \infty
\end{aligned}
$$

is finite, see e.g. c.f. [Sim79]. Exploiting the semigroup property of a transition density we see that the operators of the semi group are in fact of trace class

$$
\begin{aligned}
Tr\exp[t\mathcal{L}] &= \int u(t,x,y,x,y)\,dx\,dy = \iint u(\tfrac{t}{2},x,y,x',y')u(\tfrac{t}{2},x',y',x,y)\,dx'dy'dxdy \\
&= \langle \exp[\tfrac{t}{2}\mathcal{L}], \exp[\tfrac{t}{2}\mathcal{L}] \rangle.
\end{aligned}
$$

Inserting the expansion we derived in Section 4 we find:

Theorem 5.1. *Under the hypothesis of Proposition 3.1 we have the following asymptotic expansion with respect to a small parameter t.*

$$
Tr\exp[t\mathcal{L}] = \int_{T^*M} u(t,x,y,x,y)dx\,dy = \frac{12^{\frac{n}{2}}}{(2\pi)^n t^{2n}}(volM + Rt^3 + a_4 t^4 + \cdots + a_\nu t^\nu + O(t^{\nu+1})),
$$

where $volM$ is the volume of M and the first non trivial coefficient R is the scalar curvature. All further coefficients a_j, $j = 4, \ldots$ are invariant quantities of M which can be computed by the method indicated before.

Acknowledgements

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